Homework problem set 4

Submission deadline on 15 November 2021 at noon

Problem 1. The b-c *system* is the algebra with ("fermionic") generators b_n, c_n $(n \in \mathbb{Z})$ and relations $\{b_n, b_m\} = \{c_n, c_m\} = 0$, $\{b_n, c_m\} = \delta_{n+m}$, where $\{\cdot,\cdot\}$ is the anti-commutator.

(i) Show that the b-c system has a unique irreducible representation Λ containing a vector v_0 such that $b_n v_0 = 0$ for all $n \geq 0$ and $c_n v_0 = 0$ for all $n \geq 1$.

(ii) For $J \in \mathbb{C}$, let

$$L_n = \sum_{k\in\mathbb{Z}} (nJ - k) b_k c_{n-k} \text{ for } n \neq 0$$

Show that the L_n are well-defined on Λ, and that, together with

$$L_0 = \frac{1}{2}[L_1, L_{-1}],$$

they obey the relations of the Virasoro algebra with central charge

$$c = -2(6J^2 - 6J + 1)$$

and that $L_0 v_0 = -\frac{1}{2}J(J-1)v_0$. *Hint:* Proceed as in our calculation of the Virasoro algebra for the free boson and first calculate $[L_n, b_m] = (nJ - n - m)b_{n+m}$ and $L_n, c_m] = (-nJ - m)c_{n+m}$.

Problem 2. Consider the gauge fixed Polyakov action

$$S_p = \frac{T}{2}\int d\tau d\sigma \left((\partial_\tau X)^2 - (\partial_\sigma X)^2\right)$$

for an open string.

(i) Show by varying the action that we get two type of possible boundary conditions. For $\sigma_b \in \{0, \pi\}$ these are Neumann boundary conditions: $\partial_\sigma X^\mu|_{\sigma=\sigma_b} = 0$ and Dirichlet boundary conditions: $\delta X^\mu|_{\sigma=\sigma_b} = 0$.

(ii) We have seen the mode expansion for the string with Neumann boundary conditions at both ends (NN) in the lecture. Show that the mode expansion for a string with Dirichlet boundary conditions on both ends (DD) is

$$X^\mu(\tau,\sigma) = x_0^\mu + (x_1^\mu - x_0^\mu)\frac{\sigma}{\pi} + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n}{n} e^{-in\tau} 2\sin(n\sigma).$$

Calculate the μ-component of the momentum of a string with (DD) boundary conditions in this direction. *Hint:* $\delta X^\mu|_{\sigma=\sigma_b} = 0$ implies that $\partial_\tau X^\mu|_{\sigma=\sigma_b} = 0$. What does this imply for $\partial_\tau X_L$ and $\partial_\tau X_R$?