Homework problem set 3

Submission deadline on 8 November 2021 at noon

Problem 1 (All about the angle)**.** In the lecture we introduced a conformal transformation as a diffeomorphism under which the metric changes only by an overall factor. Consider $\mathbb{R}^n$ equipped with η the standard euclidean metric. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. In local coordinates the induced metric $\tilde{\eta} = f^*\eta$ on M can be written as

$$\tilde{\eta}_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial f^\alpha(x)}{\partial x^\mu} \frac{\partial f^\beta(x)}{\partial x^\nu}$$

By definition the diffeomorphism f is conformal if $\tilde{\eta} = f^*\eta = e^{w(x)}\eta$. For an infinitesimal diffeomorphism ζ^μ this translates to the condition:

$$\partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu = w(x)\eta_{\mu\nu} \tag{1}$$

(i) Show by appropriate contraction and differentiation of the above equation that for a conformal transformation the following identities must hold:

$$w(x) = \frac{2}{d}\partial \cdot \zeta$$
$$(\eta_{\mu\nu}\partial^2 + (d-2)\partial_\mu\partial_\nu)w(x) = 0$$
$$2\partial_\mu\partial_\nu\zeta_\rho = (-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)w(x)$$
$$(d-1)\partial^2 w(x) = 0$$

(ii) Show that for $n \geq 3$ a generator of a conformal transformation is given by

$$\zeta_\rho = a_\rho + b_{\rho\nu}x^\nu + c_{\mu\nu\rho}x^\mu x^\nu$$

It follows that in this case the algebra of conformal transformations is finite dimensional. How many degrees of freedom does it have? Discuss the physical interpretation transformations generated by the different degrees of freedom.
Optional: Find the generators of the transformation and determine the commutators. Which Lie algebra do they generate?
Hint: To count the degrees of freedom, use the identities in (i) to show that you can write $b_{\mu\nu} = \alpha \cdot \eta_{\mu\nu} + m_{\mu\nu}$ where $m_{\mu\nu} = -m_{\nu\mu}$ and $c_{\mu\nu\rho} = (-c_\rho\eta_{\mu\nu} + \eta_{\rho\mu}c_\nu + \eta_{\rho\nu}c_\mu)$ for some vector c_μ. For the interpretation of the transformations you can consider $a_\mu, \alpha, m_{\mu\nu}$ and c_ν separately.

(iii) By introducing complex coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, show that for $n = 2$, equation (1) is equivalent to the Cauchy-Riemann equations. Follow that the algebra of conformal transformations is infinite dimensional.

(iv) Convince yourself that the generators of the conformal transformations in $n = 2$ are given by $l_n = -z^{n+1}\frac{\partial}{\partial z}$. Show that they satisfy the Witt algebra

$$[l_n, l_m] = (m-n)l_{m+n}.$$

Problem 2 (Linear curvature)**.** In general relativity the curvature of a general $D = 1+d$-dimensional spacetime is described by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{\kappa^2_{(D)}}{4}T_{\mu\nu}$$

for some constant κ_D. For many physical phenomena gravity is very weak, and the metric $g_{\mu\nu}(x)$ can be chosen to be very close to the Minkowski metric

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa_{(D)}h_{\mu\nu}(x)$$

and we view $h_{\mu\nu}$ as a small fluctuation.

(i) Show that to first order in $h_{\mu\nu}$ the Einstein equations take the form

$$\partial^\sigma\partial_\sigma(h_{\mu\nu} - \eta_{\mu\nu}h^\sigma{}_\sigma) - \partial^\sigma\left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu}\right) + \partial_\mu\partial_\nu h^\sigma{}_\sigma + \eta_{\mu\nu}\partial^\rho\partial^\lambda h_{\rho\lambda} = -\frac{\kappa_{(D)}}{2}T_{\mu\nu} \tag{2}$$

where index are raised and lowered with respect to $\eta_{\alpha\beta}$. Show that by introducing $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^\sigma{}_\sigma\eta_{\mu\nu}$ you can bring (2) to the form

$$\partial^\sigma\partial_\sigma\overline{h}_{\mu\nu} - \partial^\sigma\left(\partial_\mu\overline{h}_{\sigma\nu} + \partial_\nu\overline{h}_{\sigma\mu}\right) + \eta_{\mu\nu}\partial^\rho\partial^\lambda\overline{h}_{\rho\lambda} = -\frac{\kappa_{(D)}}{2}T_{\mu\nu}$$

(ii) We have seen in the lecture that under an infinitesimal diffeomorphism ζ_μ the metric transforms as

$$\delta g_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$$

Consider a transformation ζ_μ that solves the inhomogeneous Laplace equation $\partial^\sigma\partial_\sigma\zeta_\nu = -(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h^\sigma{}_\sigma)$ and define $g'_{\mu\nu} = g_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$. This choice of gauge is called the harmonic gauge. Show that the linearized Einstein equations reduce to

$$\partial^\sigma\partial_\sigma\overline{h'}_{\mu\nu} = -\frac{\kappa_{(D)}}{2}T_{\mu\nu}$$

For in absence of a source this is just a wave equation describing gravitational waves.

(iii) Assume $d \geq 3$. Show that for a point particle of mass M at rest at the origin in space, the metric (in harmonic gauge) takes the form $\kappa h_{\mu\nu} = 2\Phi_g(\delta_{0\mu}\delta_{0\nu} + \eta_{\mu\nu}/(D-2))$ where

$$\Phi(r) = -\frac{G_{(D)}M}{r^{(d-2)}}$$

Express $G_{(D)}$ in terms of κ and determine its dimension.
Hint: Recall that contraction is always with respect to $\eta_{\mu\nu}$. Show that $\partial^\tau\partial_\tau h^\sigma{}_\sigma = \kappa_{(D)}T^\sigma{}_\sigma/(D-2)$ and notice that for a particle in rest only $T_{00} = \rho(x)$ is non-zero. We are interested in static solutions, i.e. the solution should be independent of x_0. For $d \geq 3$ the Green's function of the Laplace equation satisfying $-\Delta u(x) = \delta^{(d)}(x)$ is given by

$$u(x) = \frac{1}{d(d-2)\alpha(d)}\frac{1}{|x|^{d-2}}$$

where $\alpha(d)$ is the volume of the d-dimensional unit ball.