Homework problem set 2

Submission deadline on 1 November 2021at noon

Problem 1 (Almost flat world). (i) Verify, using a method of your choice, that under a conformal transformation, $\tilde{h} = e^{2f}h$, the Ricci scalar (scalar curvature) of a Riemannian or pseudo-Riemannian manifold (M,h) transforms as

$$\tilde{R} = e^{-2f}R - 2(n-1)e^{-2f}\Delta f - (n-2)(n-1)e^{-2f}h^{-1}(df,df)$$

where n is the dimension of M, Δ is the Laplace-Beltrami operator defined on scalar functions as $\Delta f = \nabla\cdot\nabla f$ or in coordinates as

$$\Delta f = \frac{1}{\sqrt{|det(h)|}}\partial_i\left(\sqrt{|det(h)|}h^{ij}\partial_j f\right).$$

and h^{-1} is the induced metric on covectors. *Hint:* If you use Christoffel symbols, remember that they vanish at the origin of Gaussian normal coordinates.

(ii) Deduce that when $n = 2$, in the neighborhood of any point on M, there exists a function f such that $\tilde{h}$ is flat. *Hint:* In $n = 2$, the Riemann tensor is completely determined by the Ricci scalar. You can use without proof that the resulting Poisson equation has a solution.

Problem 2 (Good vibrations). We showed in class (assuming the result of problem 1) that given any classical string solution, there are worldsheet coordinates (τ,σ) in which

$$X(\tau,\sigma) = \frac{1}{2}\big(F(\tau+\sigma) + G(\tau-\sigma)\big)$$

where $F(u)$ and $G(v)$ are functions, valued in D-dimensional Minkowksi space, satisfying $F'^2 = G'^2 = 0$. We consider only closed strings in this problem, so that $X(\tau,\sigma+\sigma_1) = X(\tau,\sigma)$ is periodic in σ.

(i) Explain why, for a solution with a reasonable time-evolution, one may assume $X^0(\tau,\sigma) = \tau$ (i.e., flat fiducial gauge and static gauge are compatible). Express the general solution in terms of two periodic functions $f(u)$ and $g(v)$ valued in the $(D-2)$-dimensional unit sphere, and appropriate integration constants.

(ii) Write down the time evolution of a string which, at time $X^0 = 0$, forms a circle of radius R at rest in the X^1-X^2-plane. Calculate the mass of this solution.

(iii) Show that, for a generic solution in $D = 4$, there are points u_*, v_* in parameter space for which $f(u_*) = g(v_*)$. Show that around such points, the trace of the string in spacetime forms a cusp singularity moving (instantaneously) at the speed of light. *Hint:* An (ordinary) cusp in the x-y-plane can be defined (locally) as the set of solutions of the equation $x^3 = y^2$. You will find the cusp in parametrized form by expanding f and g around the singular point.