Homework problem set 14

Submission deadline on 7 February 2022 at noon

Problem 1 (Supersymmetric Quantum Mechanics)**.** Consider a quantum mechanical system consting of one bosonic degree of freedom ($[p,x] = -i$ represented as usual by $p = -i\frac{d}{dx}$), and one fermionic degree of freedom ($\{\psi,\psi^\dagger\} = \psi\psi^\dagger + \psi^\dagger\psi = 1$, represented by the matrices $\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \psi^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$). If $V(x)$ is an arbitrary function of x, and $V'(x) = \frac{d}{dx}V(x)$, define the operators

$$Q = \psi(ip + V'(x)), \quad Q^\dagger = \psi^\dagger(-ip + V'(x))$$

(i) Show that $Q^2 = (Q^\dagger)^2 = 0$. Define $H := \{Q,Q^\dagger\} = QQ^\dagger + Q^\dagger Q$ and show that $[Q,H] = 0$. Q is called a supersymmetry of the Hamiltonian H. Show that $H = p^2 + (V')^2 + [\psi,\psi^\dagger]V''$.

(ii) One of the features of supersymmetry is that the zero-energy states can be determined by solving the simple first order equations $Q|\phi\rangle = 0$, $Q|\phi\rangle = 0$ (why?). By imposing appropriate boundary conditions as $x \to \pm\infty$, determine the condition on V under which H has such a supersymmetric ground state.

Problem 2 (More Supersymmetric Quantum Mechanics)**.** (*Optional)* Let M be a compact, oriented manifold of dimension n. We denote a (generic) set of coordinates of M by x^I. We consider a theory with n bosonic variables ϕ^I representing the position of the particle. These are represented by a map $\mathbb{R} \to M$ and locally $\phi^I = x^I \circ \phi$. Furthermore we have a complex fermionic field, which is represented by sections $\phi, \overline{\phi} \in \Gamma(\mathbb{R}, \phi^* TM \otimes \mathbb{C})$. Locally $\phi = \phi^I (\partial/\partial x^I)|_\phi$ and similarly for $\overline{\phi}$. Consider the Lagrangian

$$L := \frac{1}{2} g_{IJ}\dot{\phi}^I\dot{\phi}^J + \frac{i}{2} g_{IJ}(\overline{\psi}^I D_t\psi^J - D_t\overline{\psi}^I\overline{\psi}^J) - \frac{1}{2} R_{IJKL}\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L \tag{1}$$

where

$$D_t\psi^I = \partial_t\psi^I + \Gamma^I_{JK}\partial_t\phi^J\psi^K$$

with Γ^I_{JK} the Christoffel symbols of the Levi-Civita connections.

(i) Show that (1) is invariant under the supersymmetry transformations

$$\begin{aligned} \delta\phi^I &= \epsilon\overline{\psi}^I - \overline{\epsilon}\psi^I \\ \delta\psi &= \epsilon\left(i\dot{\phi}^I - \Gamma^I_{JK}\overline{\psi}^J\psi^K\right) \\ \delta\overline{\psi} &= \overline{\epsilon}\left(-i\dot{\phi}^I - \Gamma^I_{JK}\overline{\psi}^J\psi^K\right). \end{aligned}$$

Hint: Remind yourself of the properties of the Riemann tensor, in particular of the second Bianchi identity $R^\alpha{}_{\beta[\mu\nu;\sigma]} = 0$.

(ii) Calculate the Noether charges for the symmetry. Further show that there is a $U(1)$ symmetry acting on ψ and $\overline{\psi}$ and calculate the corresponding Noether charge F.

(iii) Perform canonical quantization of this system. Fixing the Hamiltonian to be

$$\{Q,Q^\dagger\} = 2H$$

show that there is a quantum supersymmetry and that F is conserved.

(iv) Identify the space

$$\mathscr{H} = \Omega^\bullet(M) \otimes \mathbb{C}$$

with inner product

$$\langle w_1, w_2 \rangle = \int_M \overline{w_1} \wedge * w_2$$

as the Hilbert space for this theory. How do the quantum operators act on this space and find an interpretation for the grading. How can one describe the space of ground states?
Hint: Consider the usual operators on the space of differential forms, like the differential and contraction. Note that you can use the inner product to define adjoint operators.

(v) The supersymmetric index is defined as $\mathrm{Tr}(-1)^F$ on the space of ground states. What is the interpretation of this index?
Hint: You can use without proof that every de Rham cohomology class $[\tilde{w}] \in H^p_{dR}(M)$ has a unique harmonic representative $w \in \Omega^\bullet(M)$, i.e. there is one and only one $w \in \Omega^\bullet(M)$ with $\Delta\omega = 0$ and $w \equiv \tilde{w} \bmod \mathrm{im(d)}$.

Problem 3 (Super Virasoro algebra)**.** Consider an infinite set of bosonic oscillators α_n (labelled by $n \in \mathbb{Z}$), together with an infinite set of (Neveu-Schwarz) fermionic oscillator β_r (labelled by $r\mathbb{Z} + 1/2$), satisfying the commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n+m}, \quad \{\beta_r, \beta_s\} = \delta_{r+s}$$

Define the basis of the super-Virasoro algebra

$$\begin{aligned}
L_n &= \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}\alpha_{n+p} + \frac{1}{2}\sum_{s\in\mathbb{Z}+\frac{1}{2}} s\beta_{-s}\beta_{n+s} \\
L_0 &= \frac{1}{2}\alpha_0\alpha_0 + \sum_{\mathbb{Z}_{>0}} \alpha_{-p}\alpha_p + \sum_{s\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} s\beta_{-s}\beta_s \\
G_r &= \sum_{p\in\mathbb{Z}} \alpha_{-p}\beta_{r+p}
\end{aligned}$$

which is normal ordered with respect to the standard representation on the product of Fock space (i.e. oscillators with negative index are creation operators).

(i) Verify the commutation relations

$$\begin{aligned}
[L_n, \alpha_m] &= -m\alpha_{n+m} \\
[L_n, \beta_r] &= \left(-\frac{n}{2} - r\right)\beta_{n+r} \\
[G_r, \alpha_n] &= -n\beta_{r+n} \\
\{G_r, \beta_s\} &= \alpha_{r+s}
\end{aligned}$$

(ii) Show that for $n \neq 0$,

$$\sum_{r\in\mathbb{Z}+\frac{1}{2}} \beta_{-r}\beta_{n+r} = 0$$

(iii) Using the results of (i), explain the evaluation of the following commutators

$$\begin{aligned}
[L_n, G_r] &= \left(\frac{n}{2} - r\right)G_{n+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \left(\frac{r^2}{2} - \frac{1}{8}\right)\delta_{r+s}.
\end{aligned}$$