Homework problem set 12

Submission deadline on 24 January 2022 at noon

Problem 1 (Narain Moduli Space)**.** It was explained in Lecture 19 that a flat metric on a k-dimensional torus $T^k = \mathbb{R}^k_y/(2\pi\mathbb{Z}^k)$ is induced by identifying $T^k = \mathbb{R}^k_x/(2\pi\Lambda)$, where $\mathbb{R}^k_x$ is equipped with the standard euclidean metric and $\Lambda = R\mathbb{Z}^k$, where $R = (R^i_\alpha)$ is an invertible $k\times k$ matrix, that a string background requires the datum of a flat B-field, which in x-coordinates can be thought of as an anti-symmetric matrix $B_{ij} = -B_{ji}$, but that to understand the moduli space it is best to replace both with the set of left/right-moving charges

$$\Gamma = \left\{\begin{pmatrix} p_L \\ p_R \end{pmatrix}\right\} = S\mathbb{Z}^{2k} \subset \mathbb{R}^k_x \oplus \mathbb{R}^k_x$$

where $S = \frac{1}{\sqrt{2}}\begin{pmatrix} R^{-T} & R-BR \\ R^{-T} & -R-BR \end{pmatrix}$, which Γ is an even and self-dual lattice with respect to the indefinite inner product $p_L^2 - p_R^2$, whence

$$\mathscr{M}_k = {}_{SO(k;\mathbb{R})\times SO(k;\mathbb{R})}\backslash^{SO(k,k;\mathbb{R})}/_{SO(k,k;\mathbb{Z})}$$

(i) Describe explicitly the change of basis that shows $S \in SO(k,k;\mathbb{R})$.

(ii) Argue that every even, self-dual lattice of signature (n,n) arises for some S as above.

(iii) Show in detail that the one-loop partiton function

$$Z_{T^k} = |\eta|^{-2k} \sum_{(p_L,p_R)\in\Gamma} e^{2\pi i\left(\tau\frac{p_R^2}{2} - \bar\tau\frac{p_L^2}{2}\right)}$$

is modular invariant. *Hint:* You may use without additional argument the general Poisson summation formula

$$\sum_{\gamma\in\Gamma} f(\gamma) = \frac{1}{|\Delta|}\sum_{\gamma^\vee\in\Gamma^\vee} \hat f(\gamma^\vee)$$

where $\Delta = \det S$, and $\hat f$ is the Fourier transform of f.

Problem 2 (Dimensional Reduction of Yang-Mills theory)**.** Consider $(p+1)$-dimensional $U(N)$ Yang-Mills theory with action

$$S_{\text{YM}} = -\frac{1}{4\big(g^{(p+1)}_{\text{YM}}\big)^2}\int d^{p+1}x\, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \qquad (1)$$

(i) Show that upon Kaluza-Klein compactification on a circle in the trivial background $A_\mu = 0$, the massless fields consist of a p-dimensional $U(N)$ gauge multiplet and an adjoint-valued scalar.

(ii) Describe what happens in a non-trivial background with constant A_p, and give a geometric interpretation in terms of Dp-branes wrapped on the circle, or D$(p-1)$-branes wrapped on the dual circle.

(iii) Find the field content and scalar potential for dimensional reduction to 1 dimension (i.e., in the trivial background).