Homework problem set 10

Submission deadline on 10 January 2022 at noon

Problem 1. In conformal gauge and complex coordinates (and with $\alpha' = 1$), the Polyakov action for the bosonic string including ghosts is

$$S = S_m + S_{gh} = \frac{1}{2\pi}\int \partial X \bar{\partial} X \, d^2z + \frac{1}{2\pi}\int \left(b\bar{\partial}c + \tilde{b}\partial\tilde{c}\right) d^2z$$

Rederive or verify as many as you can of the following statements.

(i) The components of the energy-momentum tensors are, respectively,

$$\begin{aligned} T_m &= -\partial X \partial X\,, & \bar{T}_m &= -\bar{\partial} X \bar{\partial} X \\ T_{gh} &= 2\partial c b + c\partial b\,, & \bar{T}_{gh} &= 2\bar{\partial}\tilde{c}\tilde{b} + \tilde{c}\bar{\partial}\tilde{b} \end{aligned}$$

(ii) The combined action is invariant under the odd BRST symmetry

$$\begin{aligned} &\delta X = c\partial X + \tilde{c}\bar{\partial} X \\ &\delta c = c\partial c\,, & &\delta\tilde{c} = \tilde{c}\bar{\partial}\tilde{c} \\ &\delta b = T_m + T_{gh}\,, & &\delta\tilde{b} = \bar{T}_m + \bar{T}_{gh} \end{aligned}$$

Hint: Use $d = \partial + \bar{\partial}$ and $\int d(\cdots) d^2z = 0$.

(iii) The Noether current associated with the BRST symmetry splits into holomorphic and anti-holomorphic parts given by

$$\begin{aligned} J &= cT_m + \frac{1}{2}cT_{gh} + \frac{3}{2}\partial^2 c \\ \bar{J} &= \tilde{c}\bar{T}_m + \frac{1}{2}\tilde{c}\bar{T}_{gh} + \frac{3}{2}\bar{\partial}^2\tilde{c} \end{aligned}$$

Remarks: You are welcome to reuse results from previous assignements or cross-check formulas with textbooks (such as Polchinski, Sec. 4.3), but make sure you trust yourself all signs and factors of 2.

Problem 2 (Ramanujan meets Strings)**.** Recall that for the open string the mass is quantized as

$$\alpha' M^2 = n - 1$$

We are interested in the asymptotic behaviour of the number of states at a given mass level which we denote d_m.

(i) Recall that the number operator is

$$N := \sum_{n=1}^{\infty}\sum_{n=1}^{24} \alpha_{-n}^i \alpha_n^i$$

Argue that the d_n appears as the coefficients of $\mathrm{Tr}_{\mathscr{H}_{LCQ}}(q^N)$ where q is a formal variable and show that

$$\mathrm{Tr}_{\mathscr{H}_{LCQ}}(q^N) = \prod_{n=1}^{\infty}(1-q^n)^{-24}$$

Write this in terms of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q)^n$$

where now $q = exp(2\pi i\tau)$.

(ii) As usual you can get the coefficients using a contour integral. Use the identity

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$$

to write $d(m) := d_m$ as

$$d(m) = \int d\tau \sqrt{-i\tau} \frac{q^{-m}}{\eta(-\frac{1}{\tau})}$$

Then, treating m as continuous find a function $D(m)$ such that $D'(m) = d(m)$. Approximate $D(m)$ with the method of steepest decent to show that for large m the

$$d(m) \sim exp(4\pi\sqrt{m}) \tag{1}$$

If you are very ambitious you can additionally perform the calculation with the exponent -24 replaced by -1. What is the interpretation of the coefficients for this series?

(iii) The partition function of a single string is given by

$$Z = \int_0^{\infty} dm \rho(m) exp(-m/T)$$

for $\rho(m)$ the asymptotic density of states you calculated above. Determine the highest temperature, such that this integral converges. This temperature is called to Hagedorn temperature and can be interpreted as the highest possible temperature in a stringy universe.