Lecture 16 Ghosts & BRST

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Basic point: is to give a more physical (action based) way to think about determinants and their anomalies, and a more flexible (cohomological) way to understand formal properties, invariance etc. Also opens the door to the superstring.

0 Recap

- Determinants can be represented as (Gaussian) Berezin integration over fermionic (Grassmann) variables:

- If θ is "odd variable", $\theta^2=0$, the most general function is of the form

$$f(\theta) = f_0 + \theta f_1$$

and we define $\int f(\theta)\, d\theta := f_1 \quad (\in \mathbb{C})$.

- If $\theta^1 \dots \theta^n$ are several such, $\theta^i\theta^j = -\theta^j\theta^i$,

$$\int f(\theta^1 \dots \theta^n)\, d^n\theta = \text{"coefficient of } \theta^1\cdots\theta^n\text{, beware of sign!"}$$

- In particular, if $A^k_i\theta^i$ $k=1,\dots n$ are n linear functions,

$$\int \prod_{k=1}^n A^k_i\theta^i \, d^n\theta = \det A.$$

which has two interpretations.

- Assuming A invertible, we can view it as a change of variables, to

$$\tilde{\Theta}^k = A^k_i \Theta^i$$

so that to preserve

$$\int f(\tilde{\Theta})\, d^n\tilde{\Theta} = \int A^*(f\, d^n\tilde{\Theta}) = \int f(A\Theta)\, \det A^{-1}\, d^n\Theta$$

we learn $A^*(d^n\tilde{\Theta}) = \det A^{-1} d^n\Theta$

(won't be important)

- In any case, viewing $A: V \to W$ as a linear map of odd vector spaces, with "orthonormal coordinates" Θ^i on V and η_k on $W^\vee$, we can write

$$\det A = \int \prod_{k=1}^{n} \eta_k A^k_i \Theta^i \, d^n\eta\, d^n\Theta$$

$$= \int \exp(\eta_k A^k_i \Theta^i)\, d^n\eta\, d^n\Theta.$$

$$= \int \exp A \, d\eta\, d\Theta$$ where $A \in V^\vee \otimes W$ is function on $V \otimes W^\vee$

N.B. ① • $\int f\, d\Theta = 0$ if f "does not depend on Θ", in particular $\det A = 0$ if A has a non-trivial kernel/cokernel. (In distinction to bosonic case where it would be divergent.)

- We won't care about overall sign.

① Analogue model

- In the situation $M \circlearrowleft G$ with

$$\frac{1}{\mathrm{Vol}(G)} \int_M f(x)\, d\mathrm{vol}_g = \int_T \det X \; f(t)\, d\mathrm{vol}_{g/T}$$

we introduce orthonormal odd coordinates c^α on $\mathfrak{g}[1]$ (basis of $\mathfrak{g}^\vee$, anticommuting, "ghosts") and b_j on $\mathfrak{g}_M = X(\mathfrak{g}) \subset TM$ (identified with $\mathfrak{g}_M^\vee$; antighosts) to write

$$\det X = \int dc\, db \; e^{-b_j X^j_\alpha c^\alpha} \quad \text{(up to overall sign)}$$

- $$b_j X^j_\alpha c^\alpha = \langle b, Xc\rangle_{TM} = \langle X^t b, c\rangle_{\mathfrak{g}}.$$

- Apart from general cuteness, this formulation allows for a "uniform flexible treatment" of issues related to "zero modes".

For example, if we "made a mistake" of "introducing too many antighosts", i.e. $\operatorname{coker} X \neq 0$, then the corresponding subspace of TM really belongs to T and we can correct for this part of the determinant by including

$$\prod_k b_j \frac{\partial y^j}{\partial t^k}$$

into the integral.

- Conversely if $\operatorname{Ker} X \neq 0$, we can enlarge M by some space Σ of the right dimension on which G acts effectively by vector fields Y^i_α and include

$$\prod_\alpha c^\alpha Y^i_\alpha$$

into the integral.

- But perhaps most interestingly, for the physical model, if $f(x) = e^{-S_m}$, the "combined action"

$$S_m + S_{gh} \qquad S_{gh} = b_j X^j_\alpha c^\alpha$$

is invariant under odd "BRST symmetry"

$$\delta c^\alpha = \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \qquad f^\alpha_{\beta\gamma} \text{ : structure constants of } \mathfrak{g}$$

$$\delta b_j = b_i \, \partial_j X^i_\alpha c^\alpha$$

$$\delta t^k = 0$$

— really $\delta S_m = 0$ separately, while

$$\delta S_{gh} = b_i \, \partial_j X^i_\beta c^\beta X^j_\gamma c^\gamma - \frac{1}{2} b_j X^j_\alpha f^\alpha_{\beta\gamma} c^\beta c^\gamma$$

$$= 0$$

because $[X_\alpha, X_\beta] = f^\gamma_{\alpha\beta} X_\gamma \in \mathcal{X}(M)$ generate action of G.

$\delta^2 = 0$ for the same reason, see Lecture 9

② Ghosts in string theory

On account of the fact that $T\,\mathrm{Weyl} \to C^\infty(\Sigma)\, h_{\alpha\beta} \subset T\,\mathrm{Met}(\Sigma)$ is the identity w/ trivial determinant, we introduce only reparametrization

ghosts $c^\alpha(\sigma)$ – thought of as odd orthonormal coordinates on $\mathcal{H}(\Sigma)$, fermionic versions of $\zeta^\alpha(\sigma)$.

antighosts $b_{\alpha\beta}(\sigma)$ – 〃 〃 〃
on $\mathrm{Sym}^2_0(T^*\Sigma)$ $\qquad$ $\theta^0_{\alpha\beta}(\sigma)$

to ~~calculate~~ represent det P as fermionic path-integral with action
$\qquad$ 〃 $(\det P^\dagger P)^{1/2}$

$$S_{gh} = \frac{1}{2\pi} \int_\Sigma \sqrt{h}\; b_{\alpha\beta} \nabla_\gamma c^\beta\, h^{\beta\gamma}\, d^2\sigma$$

(Diff + Weyl-invariant)

which in conformal gauge and complex coordinates becomes

$$S_{gh} = \frac{1}{2\pi} \int_\Sigma d^2 z \left(b_{zz} \bar\partial c^z + \tilde b_{\bar z \bar z}\, \partial \tilde c^{\bar z} \right)$$

~~Incorporating Ker $P^\dagger$~~ Identifying Ker $P^\dagger$ with $T\mathcal{M}_g$ and fixing CKG as described before then gives the "beautiful formula" for string amplitudes in gauge fixed /BRST form.

$K = \dim \mathrm{CKG} = \dim \ker P$

$$A(1,\dots,n) = \sum_g g_s^{2g+n-2} \int_{\mathcal{M}_{g,n}} \int_{\mathrm{Map}(\Sigma,\mathbb{R}^{1,d}) \times \mathrm{Sym}_0^2(T^2\Sigma)[1] \times \mathcal{H}(\Sigma)[1]} e^{-S_p - S_{gh}}$$

$$\prod_{i=1}^{n-K} V_i(z_i,\bar z_i) \prod_{j=n-K+1}^{n} c(z_j)\tilde c(\bar z_j) \sqrt{\det \hat h(z_j,\bar z_j)}\, V_j(z_j,\bar z_j) \prod_{k=1}^{\dim \mathcal{M}_g} (b,\mu_k)(\hat b, \bar\mu_k)$$

$$\mathcal{D}c\, \mathcal{D}\tilde c\, \mathcal{D}b\, \mathcal{D}\hat b\, \mathcal{D}X \prod_{i=1}^{n-K} d^2 z_i \prod_{k=1}^{\dim \mathcal{M}_g} dt^k\, d\bar t^k .$$

Remarks: . This formula does not change anything in practical calculations (which are rarely carried out for $g>1$).
- Apart from general cuteness, treating c, b as worldsheet fields makes the local character of the gauge fixing manifest, gives a physicsy way of understanding anomalies, and allows for "easy" proofs of formal properties of A (such as: unitarity, finiteness).

In particular:
- b,c - system defined by S_{gh} is (non-unitary) CFT with central charge -26 (cmp. ghost module Λ from lecture 10 & exercises)
  $\rightsquigarrow$ Virasoro and Weyl anomaly are on equal footing with matter sector.
- The (non-unitarity of) b-c system cancels against (that of) the ("unphysical") light-cone coordinates, with a transition from global (path-integral) to local (Hilbert space, with constraints from large residual gauge group) considerations mediated by BRST symmetry.

3 BRST symmetry (INCOMPLETE!)

In a way that I did not fully work out in f.d. model, the Diff ($\ltimes$ Weyl)-invariance of S_P is thought of as arising in two steps:

(i) Lack of diffeomorphism invariance of S_P, as manifested by our considerations in Lecture 4

$$\delta_\xi S_P\Big|_{h_{\alpha\beta}=\text{const}} = \int \nabla^\alpha \xi^\beta \, T^{\text{can.}}_{\alpha\beta}$$

(ii) Compensation by variation of "gauge field $h_{\alpha\beta}$",

$$\delta S_P = \int \delta h^{\alpha\beta} \, T^{E\text{-}H}_{\alpha\beta}$$

$$\delta h^{\alpha\beta} = \nabla^\alpha \xi^\beta , \quad T^{\text{can}}_{\alpha\beta} = T^{E\text{-}H}_{\alpha\beta}$$

(up to total derivative)

· roughly speaking, there is a non-zero "partial" derivative $\frac{\partial S}{\partial \xi^\alpha}$ that gets compensated (before gauge fixing) by $\frac{\partial S}{\partial X^i}\frac{\partial X^i}{\partial \xi^\alpha}$ and (can be taken to) contribute additional term to BRST invariance after gauge fixing.

In the case at hand,

$$S'_{tot} = S_P + S_{gh}$$

$$= \frac{1}{4\pi\alpha'}\int\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X + \frac{1}{2\pi}\int\sqrt{h}\,b_{\alpha\beta}\nabla^\alpha c^\beta$$

is invariant under BRST symmetry.

$$\delta c^\alpha = c^\beta\nabla_\beta c^\alpha$$

$$\delta b_{\alpha\beta} = c^\gamma\nabla_\gamma b_{\alpha\beta} + T^{gh}_{\alpha\beta} + T^{M}_{\alpha\beta}$$

$$\delta X = c^\alpha\nabla_\alpha X$$

$\therefore$

$$\delta S_{tot} = \int\nabla^\alpha c^\beta T^{tot}_{\alpha\beta} + \int\left(T^{gh}_{\alpha\beta} + T^{M}_{\alpha\beta}\right)\nabla^\alpha c^\beta$$

$$= 0$$

(modulo signs and factors).

More often than not, one will find a ~~two~~ different version in which first term in variation of $b_{\alpha\beta}$ is absent and δc^α is modified in compensation ($\rightarrow$ homework).