Lecture 15 Gauge fixing 9.12.21

In the previous lecture, we have seen how a perturbative string amplitude, formally defined as

$$\int \frac{Dh\, DX}{\text{Diff} \times \text{Weyl}} e^{-S_P(X,h)} \prod V_i$$

can be "reduced" to the integral of a correlator in 2-dimensional CFT over a finite-dimensional space, where:

- "reduced" is in quotes because the procedure can not be fully justified without precise definition of p.i. $\int$ in the first place.
- the 2d CFT correlators can be defined by Gaussian integral (over an ∞-dim. space), as solutions of differential equations, or by algebraic methods.
- the end result is a totally well-defined (if complicated, transcendental) function of external momenta.

→ This is a general story. In particular, as one progresses (to superstring etc.) the rigorous def. of p.i. takes back stage and one (is often forced to) take(s) the f.d. integrals as definition (and they are hard to evaluate). But still, the p.i. picture is useful intuition building.

Purpose today: (i) what mathematics underlies this "reduction"

(ii) "physical" implementation according Faddeev-Popov/ BRST / Polyakov.

(iii) implications.

Note: · discussion abstract / formal, borderline poetic. Next explicit calculation: some loop amplitudes.
· we have reached the end of the alphabet, will not resolve notational conflicts.

① Organisation of string perturbation theory

· In (theoretical) physics in general, it is very important to assume existence of small parameters and to understand nature of perturbative expansion. In string theory, as mentioned before, there is a twofold ~~story, and both~~ and both kinds of deformations are innate to free string propagation in Minkowski space.

1) modify the quadratic Polyakov action / replace $\bigotimes_{I=2}^{d} \mathcal{F}^I$ with $\mathcal{V}^\perp_{c=24}$.

2) include string interactions.

I have already explained how 2) is contained in free theory (operator-state correspondence). What's missing is the small parameter (string coupling constant) and anything about 1).

ad 1 ∞ replacing S_P with "σ-models with pseudo-Riemannian target (M,g)" amounts to studying

$$S_g(X:\Sigma\to M, h) = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, g_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu .$$

pullback g to Σ, compare with h.

- For example, $M = \mathbb{R}^{1,d}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and $h_{\mu\nu}$ small, expanded in Fourier modes,

$$h_{\mu\nu} = \varepsilon_{\mu\nu}\, e^{ip\cdot x} \qquad \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}\ ;\ \varepsilon_{\mu}{}^{\mu} = 0.$$

- Then, $S_g = S_p + I$ where $I = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \varepsilon_{\mu\nu}\, e^{ipx}$ is small interaction which we can treat in perturbation theory (conformal).

$$\langle \dots \rangle_g = \langle \dots e^{-I} \dots \rangle_\eta \qquad \left(\text{and } p^\mu \varepsilon_{\mu\nu} = 0\right)$$

- Remarkably, I is integrated vertex operator for graviton (in flat space) and is conformally invariant (to first order) precisely if $p^2=0$, i.e. graviton is on-shell (satisfies linearized Einstein equations). (alternatively, we can think of coherent state or graviton condensate, if this mean anything to you).

- In general:
  - this persists at the non-linear level – the β-function of the non-linear σ-model is Einstein equation in target space. (*)
  - also works for other states / worldsheet operators / spacetime fields; in particular, massless states.

$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} \quad \text{non-linearises to}$$

$$S_B = i\int B_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, d^2\sigma = i\int_\Sigma X^*(B)$$

Weyl + diffeomorphism invariant b/c of antisymmetry of $B_{\mu\nu} = -B_{\nu\mu}$

$B_{\mu\nu}(X)$

(*) + higher-derivative corrections. The relevant expansion parameter is $\frac{\sqrt{\alpha'}}{R}$, where R is "typical" spacetime curvature radius (α'-expansion).

→ we will study impact of B-field on string dynamics a bit later.

· The coupling of the worldsheet theory to the massless closed string scalar (the dilaton) is given by

$$S_\Phi = \frac{1}{4\pi} \int_\Sigma \Phi(X)\, R^{(2)} \sqrt{h}\, d^2\sigma$$

where $R^{(2)}$ is the two-dimensional scalar curvature (a.k.a. Gauss curvature).

This term is more difficult to justify via vertex operators, quantum-mechanically also mixes with the Weyl anomaly. What's important for us:

If we consider a constant dilaton (which will solve eqns) $\Phi = \Phi_0 = \text{const.}$, then since locally on Σ,

$$h_{\alpha\beta} = e^{+2\varphi} \delta_{\alpha\beta}$$

we have $R^{(2)} = -2e^{-2\varphi} \partial_\alpha \partial^\alpha \varphi$ ~~and therefore~~

so S_{Φ_0} is (locally!) a total derivative, and therefore does not contribute to equations of motion (both 2-d gravity is "trivial").

· S_{Φ_0} is globally diffeomorphism invariant, and since, more generally

$$R^{(2)}_{e^{2\varphi}h} = R^{(2)}_{h} - 2e^{-2\varphi}\, \Delta_h \varphi$$

also globally Weyl invariant. (assuming Σ is compact w/o boundaries).

· It does not however, vanish globally in general. Rather, by the celebrated so-called Gauß-Bonnet theorem, we have

$$\frac{1}{4\pi}\int_{\Sigma} R^{(2)}_h \, dvol_h = \underbrace{\chi(\Sigma)}_{\substack{\text{Euler character} \\ \text{(topological invariant)}}} = 2-2\underbrace{g}_{\text{genus}(\Sigma)}$$

e.g. · $\Sigma = S^2$ (round) sphere of radius R $\quad R^{(2)} = \frac{2}{R^2}$

$vol = 4\pi R^2. \qquad g=0.$

· $\Sigma = T^2$ flat 2-torus $S^1 \times S^1$. $=0$.

·  genus 2. curvature negative.

ad 2. Claim: The purpose in life of this term is to control the other ~~deformati~~ ~~deformation,~~ ~~string~~ perturbation, string interactions.

Note: On a surface with boundary, S_{Φ_0} is ~~not~~ Weyl invariant only after adding

$$\frac{1}{2\pi}\int_{\partial\Sigma} k \, ds$$

Namely, worldsheets of progressively more complicated topology are weighted by

$$e^{-\Phi_0 \chi(\Sigma)} = e^{(2g-2)\Phi_0} = g_s^{2g-2}.$$

in the "string loop expansion". $A \sim \sum_g A_g \, g_s^{2g}$

<u>Note:</u> On a surface with boundary S'_{Φ_0} is Weyl invariant only after adding

$$\frac{1}{2\pi} \int_{\partial\Sigma} k \, ds$$

where k is "geodesic or extrinsic curvature" of $\partial\Sigma \subset (\Sigma, h)$, normalized s.t. for a circle in the plane

$$\frac{1}{2\pi} \int_{S^1} k \, ds = -1$$

$\ulcorner$ <u>Aside:</u> The open string couples to gauge field (corresponding to $\alpha^\mu_{-1}|p\rangle$ states $p^2=0$) via

$$\int_{\partial\Sigma} A^\mu \partial_\tau X^\mu \, d\tau = \int_{\partial\Sigma} X^*(A). \quad \lrcorner$$

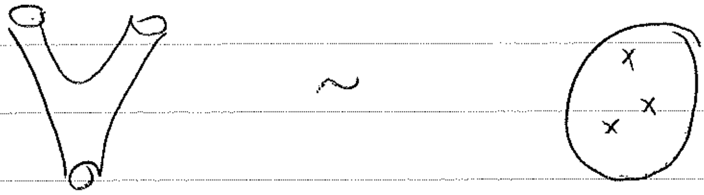
This has a number of consequences.

$$\frac{1}{4\pi} \int_\Sigma R^{(2)} \, dvol_h + \frac{1}{2\pi} \int_{\partial\Sigma} k \, ds = 2 - 2g - h$$

genus, number of boundaries

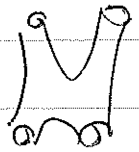
Full Gauß-Bonnet.

· Closed string vertex operators, which were obtained by mapping physical states at ∞ far away boundary to the plane, should be normalized to include a factor g_s, such that e.g.

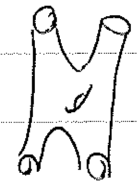


Sphere with 3 punctures

$\propto g_s^{-(2-3)} = g_s$ – the closed string coupling constant.

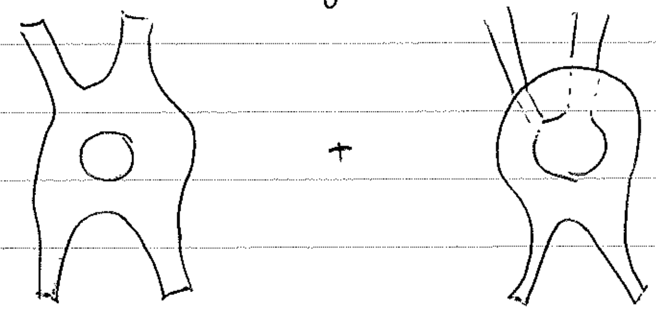


$\sim g_s^2$ (compare field theory)



$\sim g_s^4$.

· open string loops are weighted by g_s (though in general, there are already a lot more topologies to sum over.)



· open string vertex operators (which are born with deficit angle of π) are normalized with

$g_s^{1/2}$ = open string coupling constant.

~~gen~~ Summary (genus, # boundary, bulk punctures, bdry punctures)

$= (g, h, n, m)$

is weighted by $g_s^{2g+h+n+\frac{m}{2}-2}$

closed string	coupling constant	g_s
	loop counting	g_s^2
open string	coupling constant	$g_s^{1/2}$
	loop counting	g_s.

String theory has only one (dimensionless) coupling constant, which however can be understood as vev of a ~~sca~~ (dynamical) scalar field.

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② ~~M~~ Metrics and moduli of Riemann surfaces

- In the above discussion, we have tacitly assumed the fact: Compact 2-dimensional o.mfs. are classified up to diffeomorphisms by 2 non-negative integers g & h.
- There is a gabazillion ways ~~of~~ to think about this, none of which will be particularly important.

Aside: Non-orientable surfaces need in addition a third integer c the number of crosscaps, subject to equivalence

$(g, h, c) \sim (g-1, h, c+2)$ if $c > 0$

$\chi = 2 - 2g - h - c$.

- To evaluate amplitudes at each loop order χ, we rather need a good understanding of $\mathrm{Met}(\Sigma)$, and the action of $\mathrm{Diff}(\Sigma)$ and Weyl on it.

- We already know: any 2-d metric is locally conformally flat, but globally this cannot be true (in general) because of Gauß-Bonnet.

- To understand the mismatch: If (x,y) and (x',y') are local coordinates on (Σ, h) wrt. which metric has components

$$h_{\alpha\beta} = e^{2\varphi}\delta_{\alpha\beta} \qquad h'_{\alpha'\beta'} = e^{2\varphi'} \delta_{\alpha'\beta'}$$

and that are related on overlap by ~~local~~ diffeomorphism

$$(x,y) = F(x',y')$$

then $h = F^*(h')$ implies that F is local conformal transformation, namely a holomorphic map

$$z = x+iy = F(z' = x'+iy')$$

and
$$\left|\frac{dF}{dz}\right|^2 = e^{2\varphi - 2 F^*(\varphi')}.$$

Namely, given any ~~metric, we~~ fixed Riemannian metric on Σ, the class of coordinates in which h is conformal to euclidean metric defines a class of ~~local~~ complex coordinate system with holomorphic transition function, i.e. a complex structure on Σ.

<u>Fact:</u> • A 2 dimensional manifold with complex structure is called a Riemann surface. The space of $\mathcal{M}_{g,h}$ complex structures on fixed topology is (among the most studied spaces in mathematics) finite-dimensional of dimension

(g,h)	$\dim_{\mathbb{R}} \mathcal{M}_{g,h}$
$2-2g-h>0$	0
$(1,0)$	2
$(0,2)$	1
$2-2g-h<0$	$6g+3h-6$

<u>Now to action of $\mathrm{Diff}(\Sigma) \ltimes \mathrm{Weyl}$</u>. By definition, global diffeomorphisms and Weyl rescalings do not change the complex structure.
Most diffeomorphisms and all Weyl rescalings (except trivial ones) change the representative metric, except in low χ we have CKG:
conformal Killing gp

(g,h)	CKG	$\dim \mathbb{R}$
$(0,0)$	$SL(2,\mathbb{C})$	6
$(0,1)$	$SL(2,\mathbb{R})$	3
$(1,0)$	T^2	2
$(0,1)$	S^1	1
$\chi<0$	ϕ	0

<u>Relevant aspect:</u> Global structure of gauge group.

- Weyl $\cong C^\infty_+(\Sigma)$ under pointwise multiplication.
  (pos. functions)
  or even better in terms of Lie algebra $C^\infty(\Sigma)$
  $\ni \varphi \mapsto (h \mapsto e^{2\varphi} h)$.

- $\mathrm{Diff}(\Sigma)$ more complicated to describe - A large chunk is generated by vector fields on Σ, but there are also "large" diffeomorphisms that are not continuously connected to identity.
  $\rightsquigarrow$ so-called mapping class group
  in general not easy to describe.

$\mathrm{Diff}_0(\Sigma) \to \mathrm{Diff}(\Sigma) \longrightarrow MCG.$

Weyl $\to$ $\to$ Diff.

<u>Useful to know/observe</u>

- $\mathrm{Diff}(\Sigma)$ and Weyl do not commute

$$\Phi^{-1} e^{2\varphi} \Phi = e^{2\Phi^*(\varphi)}.$$

- $$\mathrm{Met}(\Sigma) / \mathrm{Diff}_0(\Sigma) \ltimes \mathrm{Weyl} = \mathcal{T}_{gh}$$

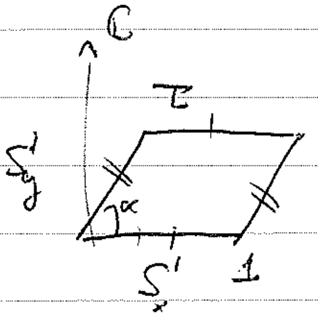
- Teichmüller space is <u>affine</u> space of dim = dim $\mathcal{M}_{g,h}$

$\mathcal{M}_{gh} = \mathcal{T}_{gh} / MCG.$

: adding punctures increases ~~reduces~~ dim $\mathcal{M}$ / reduces MCG.

Example: $\mathbb{P}^1$ done, $\mathbb{D} \cong \overline{\mathbb{H}}$ also.

$(g,h) = (1,0)$: Flat metrics on $T^2 = S^1 \times S^1$ up to constant rescaling come in 2-dim. family, parametrized by angle and ratio of radii. - Hard to visualize on donut, but easy in (complex) plane



$R_x = 1 \qquad R_y = |\tau|$

$\Sigma = \mathbb{C}/_{\mathbb{Z} \oplus \tau\mathbb{Z}}$

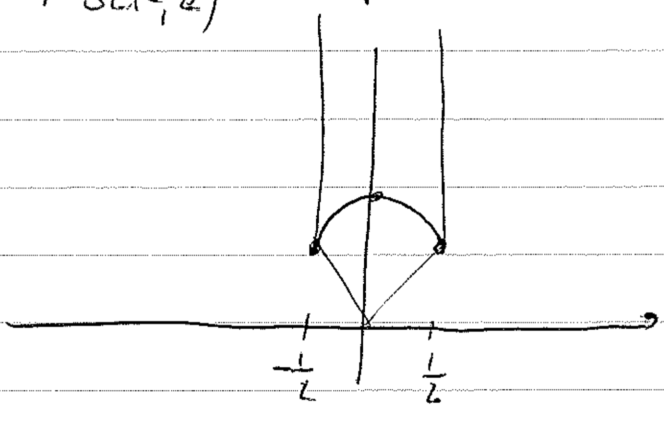
$$\tau = |R_y| \cdot e^{i\alpha} = \tau_1 + i\tau_2 \qquad \tau_2 = \operatorname{Im}\tau > 0$$

$$\rightsquigarrow \quad \mathcal{T}_{(1,0)} = \mathbb{H} = \{\operatorname{Im}\tau > 0\}$$ (do not confuse with disc)

CKG = in fact isometries, rotations $\cong T^2$.

$$\text{MCG} = SL(2,\mathbb{Z}) \qquad \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

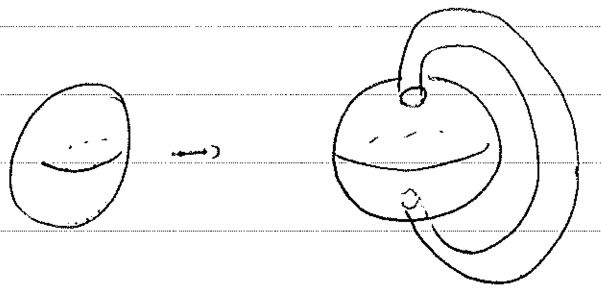
$$\mathcal{M}_{1,0} = \mathcal{T}_{1,0} / SL(2,\mathbb{Z})$$, fundamental domain



Intuition:

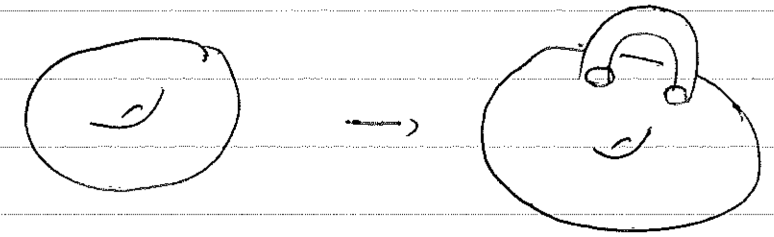
(genus, dim CKV)

(0,3)



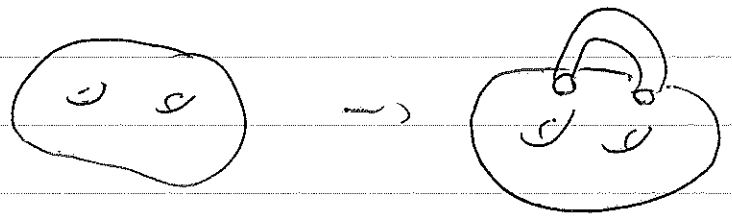
remove 2 complex CKV, add 1 complex parameter (length and twist.

(1,1)



remove 1 CKV, add 2 complex parameters (pos. of 2^{nd} pt + twist)

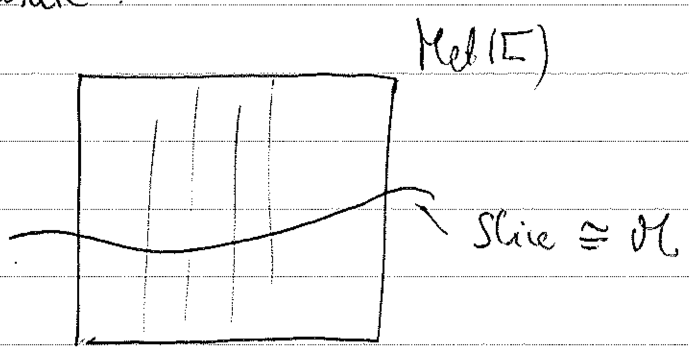
(3,0)



add 3 pars.

etc.

Main idea: ~~at least for~~ We can choose a unique Diff $\ltimes$ Weyl representative metric in each conformal class (e.g. constant curvature and fixed volume). On this representative we (work in local complex coordinates on Σ to) evaluate correlators and then integrate result over $\mathcal{M}_{g,h}$. Question again: what is the measure?



3) A finite-dimensional analogue model

is useful to understand geometric significance of Faddeev-Popov determinant, ghost and even BRST invariance, comes in two steps.

1st Lemma: let (M,g) and (N,h) be two Riemannian manifolds and $\Phi: N \to M$ a diffeomorphism, $f \in C^\infty(M)$. Then

$$\int_M f(x)\, dvol_g = \int_N f(\Phi(y))\, \mathcal{D}\, dvol_h$$

where $\mathcal{D} = (\det D\Phi^+ D\Phi)^{1/2}$ is to be explained.

Pf: $\int_{\Phi(N)} \omega = \int_N \Phi^*(\omega)$ so what we're claiming is relation between $\Phi^* dvol_g$ and $dvol_h$. In local coordinates, (x^i) on M, (y^j) on N:

$$\Phi^* dvol_g = \sqrt{\det g_{ij}}\, \det \frac{\partial \Phi^i}{\partial y^j}\, d^n y$$

$$= \sqrt{\det D\Phi^T g\, D\Phi}\, d^n y$$

~~and $D\Phi^T g$ is matrix representation of $D\Phi^+$ where adjoint is defined wrt g & h on TM and TN: $v \in TM$, $w \in TN$:~~

$$\cancel{\langle v, D\Phi(w)\rangle_g = v^T g\, D\Phi}$$

On the other hand the matrix representation of $D\Phi^{+}$ adjoint of $D\Phi$ defined wrt g, h on $TM \ni v$ and $TN \ni w$ respectively via

$$\langle v, D\Phi w\rangle_g = v^T g\, D\Phi w = (h^{-1} D\Phi^T g v)^T h w$$

$$= \langle D\Phi^{+} v, w\rangle_h$$

is $h^{-1} D\Phi^T g$ so that

$$= \sqrt{\det D\Phi^T g D\Phi}\, d^n y = (\det D\Phi^{+} D\Phi)^{1/2} \underbrace{(\det h_{ij})^{1/2} d^n y}_{dvol_h}.$$

The point being that $\det D\Phi$ is not well-defined since $D\Phi$ is operating between <u>different</u> vector spaces. (*)

<u>2^{nd}</u>: Say $M \circlearrowleft G$ acts by isometries nicely so $M/G = T$ is nice and $f(gx) = f(x)$.
Then our idea is to evaluate $\int_M f(x)\, dvol_g$ after choosing a section $s : T \to M$ which locally on T defines a diffeomorphism

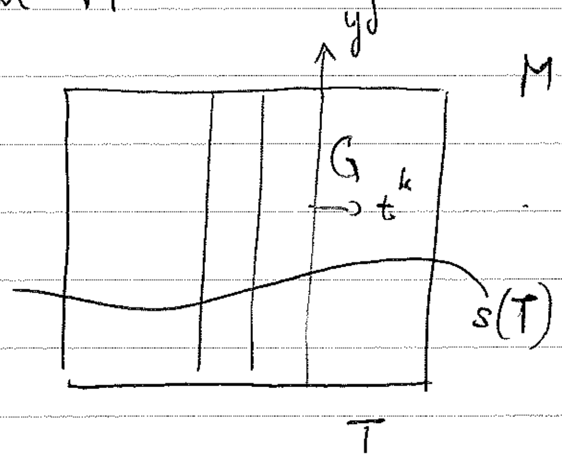
$$\Phi : G \times T \xrightarrow{id \times s} G \times M \xrightarrow[\text{action}]{gp} M$$

by pulling back to $G \times T$ where G is equipped with some "natural" (and invariant) metric k, while T with ~~invari~~ induced metric as subsmf. of M. $g|_T$.

(*) Note however that if we choose ONB on both TM and TN then $g = h = \mathrm{id}$ as matrices and $D\Phi^\dagger = D\Phi^T$ and

$$\left(\det D\Phi^\dagger D\Phi\right)^{1/2} = \left|\det D\Phi\right|$$

Letting (ξ^α) be coordinates on G, (t^k) coordinates on T and $(y^j, t^k) = (x^i)$ be "some other" adapted coordinates on M



$D\Phi$ has block form

$$D\Phi = \begin{pmatrix} \dfrac{\partial y^j}{\partial \xi^\alpha} & \dfrac{\partial y^j}{\partial t^k} \\ 0 & \dfrac{\partial t^l}{\partial t^k} \end{pmatrix} = \begin{pmatrix} X & x \\ 0 & 1 \end{pmatrix}$$

in which $X = \dfrac{\partial y^j}{\partial \xi^\alpha} = X^j_\alpha$ is the Lie map representation of

$\mathfrak{g} \longrightarrow \cancel{\mathcal{T}(M)}\ \mathfrak{X}(M)$

Under our assumptions, we may identify $X(\mathfrak{g}) \subset TM$ with a copy $\mathfrak{g}_M \subset TM|_T$ isomorphic to $\mathfrak{g}$, but with a metric induced from M.

<u>Fact:</u> Choosing ONBs in $\mathfrak{g} \times \mathfrak{g}_M$, wrt. these different metrics,

$$\det X = (\det X^{+}X)^{1/2}$$

is what is usually called Faddeev-Popov determinant.

As a result, we may write

$$\int_M f(x)\, d\mathrm{vol}_g = \int_{G \times T} f(t)\, (\det X^{+}X)^{1/2}\, d\mathrm{vol}_k\, d\mathrm{vol}_{g|_T}$$

$$= \int_G d\mathrm{vol}_k \cdot \int_T (\det X^{+}X)^{1/2} f(t)\, d\mathrm{vol}_{g|_T}$$

- If G is compact, $\int_G d\mathrm{vol}_k = \mathrm{vol}(G) < \infty$ is just overall norm. factor.
- Otherwise we may define $\int_{\tilde{M}} f(x) \frac{d\mathrm{vol}_g}{G} = \int_T \det(X^{+}X)^{1/2} f(t)\, d\mathrm{vol}_{g|_T}$
- The important pt being that $(\det X^{+}X)^{1/2} \cdot \det_{g|s(T)}$ is independent of s.

4 Situation of interest

- At the level of spaces, the analogy is

$$M = \text{fields space} = \mathrm{Met}(\Sigma) \times \mathrm{Map}(\Sigma, \mathbb{R}^{1,d})$$

$$G = \text{gauge group} = \mathrm{Diff}(\Sigma) \ltimes \mathrm{Weyl}$$

with $(F, e^{2\varphi})(h, X) = (e^{2\varphi} F^*(h),\ F^*(X))$

- Compared with "standard" derivations of FP determinant, the relevance of Riemannian metrics might seem a bit unusual. In fact however, one such is already implicit in the Gaussian integral analogy

$$G(x,y) = \text{propagator} \longleftrightarrow \int e^{-x^T A x} x^i x^j\, d^n x = \frac{A^{ij}}{\sqrt{\det A}} (2\pi)^{n/2}$$

Lebesgue measure on Euclidean space.

- Specifically, $\mathrm{Map}(\Sigma, \mathbb{R}^{1,d})$ (being a vector space) has tangent space (itself) $\mathrm{Map}(\Sigma, \mathbb{R}^{1,d})$ and on this the inner product

$$\langle X, Y \rangle = \int_\Sigma X(\sigma)\, Y(\sigma)\, d\mathrm{vol}_h$$

is the one appropriate to make $A = -\Delta_h$ self-adjoint, as is apparent from action

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \sqrt{h}\, h^{\alpha\beta} \partial_\alpha X \partial_\beta X\, d^2\sigma = \frac{1}{4\pi\alpha'} \int_\Sigma X(-\Delta_h X)\, d\mathrm{vol}_h$$

- Interestingly, the inner product on $\mathrm{Map}(\Sigma, \mathbb{R}^{1,d})$ is $\mathrm{Diff}(\Sigma)$- but not Weyl-invariant. Because however the action and classical eqs. of motion etc. are, this shows up in correlators as a "mere anomaly".
- It "comes entirely from $\det(-\Delta_h)$" and "does not depend on insertions" (indeed we hadn't seen it in prop.) With a precise definition of determinant (ζ-function regularization, $\det = \exp(-\zeta'(0))$, careful treatment of zero modes), one can show

$$\frac{1}{\det(-\Delta_{e^{2\varphi}h})^{1/2}} = \frac{1}{\det(-\Delta_h)^{1/2}}\, e^{S_L(\varphi, h)}$$

where $S_L(\varphi, h)$ is "Liouville action"

$$S_L = \frac{1}{12\pi}\int_\Sigma d\mathrm{vol}_h \left(\frac{1}{2} h^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\varphi + R^{(2)}_h \varphi\right)$$

where the connection to 2-d gravity, spacetime dilaton coupling etc. is both apparent and can't be discussed here.

- Nevertheless, the anomaly must be cancelled, by measure on $\mathrm{Met}(\Sigma)$, to give a sensible theory. (FP Determinant)
- On that space, the "correct" metric is less clear, but given that

$$T\,\mathrm{Met}(\Sigma) = \Gamma(\mathrm{Sym}^2 T^*\Sigma) \ni \Theta_{\alpha\beta} = \Theta_{\beta\alpha}\ (\sigma)$$

it is natural to instead

$$\langle \Theta, \Theta \rangle = \int_\Sigma \sqrt{h}\; \Theta^{\alpha\beta}\Theta_{\alpha\beta}\, d^2\sigma$$

$$\Theta_{\alpha\beta}\Theta_{\gamma\delta}\, h^{\alpha\gamma}h^{\beta\delta}$$

Finally, on

$$T\mathrm{Diff}(\Sigma) = \Gamma(\mathfrak{X}(\Sigma)) \ni \xi^\alpha$$

we use

$$\langle \xi, \xi \rangle = \int_\Sigma \sqrt{h}\, \xi^\alpha \xi_\alpha\, d^2\sigma$$

while on

$$T\mathrm{Weyl} = C^\infty(\Sigma) \ni \varphi$$

$$\langle \varphi, \varphi \rangle = \int_\Sigma \sqrt{h}\, \varphi^2\, d^2\sigma$$

Note that none of the above are Weyl-invariant but this don't bother us b/c we still want to cancel Weyl anomaly of X-theory. More precisely, on Weyl we could induce something from fiducial metric, but let's do the details some other time. Depart from f.d. model!

Concentrating then on the $\mathrm{Met}(\Sigma)$ factor of M, the Lie map

$$TG \longrightarrow T\mathrm{Met}(\Sigma) \quad \text{is given by}$$

$$\delta h_{\alpha\beta} = X(\xi, \varphi) = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + 2\varphi h_{\alpha\beta}$$

which however is <u>not</u> an isomorphism in distinction to f.d. model.

· The kernel

$$\mathrm{Ker}(X) = \left[(\xi,\varphi),\ \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = -2\varphi h_{\alpha\beta} \right]$$

are none other than the conformal Killing vectors.
They can be moved into $\mathfrak{X}(\Sigma)$ by separating

$$T\mathrm{Met}(\Sigma) = \underset{\substack{|\\ \text{symmetric traceless}\\ \text{2-tensors}}}{\mathrm{Sym}^2_0(T^*\Sigma)} \oplus C^\infty(\Sigma)\, h_{\alpha\beta} \qquad (*)$$

$$\tilde X(\xi,\varphi) = P(\xi)_{\alpha\beta} + (2\varphi + \nabla^\alpha\xi_\alpha)\, h_{\alpha\beta}$$

where $P(\xi)_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha - \nabla^\alpha\xi_\alpha\, h_{\alpha\beta}$

(properly speaking, this corresponds to a change of presentation of G, let's not worry.)
The point being that $(*)$ is orthogonal w.r.t. inner product on $T\mathrm{Met}(\Sigma)$:
If $\Theta^\circ_{\alpha\beta} \in \mathrm{Sym}^2_0$ then $\Theta^\circ_{\alpha\beta} h^{\alpha\beta} = 0$ and therefore

$$\langle \Theta^\circ_{\alpha\beta},\ \tilde\varphi h_{\alpha\beta} \rangle = 0 \quad \forall \tilde\varphi \text{ and conversely.}$$

· The cokernel of $X, \tilde X$ which was TT in finite-dimensional model, are those deformations of the metric that cannot be reached by diffeomorphisms or Weyl rescalings.
Namely they are tangent to Teichmüller / moduli space of Riemann surfaces. We'll use $T\mathcal{T}$.

- Although not strictly necessary, it's possible, natural and convenient to give an explicit identification of $T\mathcal{T} \subset \mathrm{Sym}^2(T^\vee\Sigma)$.
- Since $T\mathrm{Weyl} \to C^\infty(\Sigma) h_{\alpha\beta}$ is surjective, we can fix representatives of $\mathrm{Coker}\, X$ that do not change the trace, i.e. take $T\mathcal{T} \subset \mathrm{Sym}^2_0$.
- Then, using $\langle\cdot,\cdot\rangle$, we can identify

$$\mathrm{Coker}\, P = \mathrm{Im}\, P^\perp = \mathrm{Ker}\, P^\dagger$$

where $P^\dagger$ is defined by the formula

$$\langle P\xi, \Theta^0_{\alpha\beta}\rangle_{\mathrm{Sym}^2_0(T^\vee\Sigma)} = \langle \xi, P^\dagger\Theta^0_{\alpha\beta}\rangle_{\mathcal{V}(\Sigma)}$$

i.e. (up to a sign)

$$(P^\dagger\Theta^0_{\alpha\beta})^\beta = \nabla^\alpha\Theta^{0\,\beta}_\alpha \qquad \left(\text{where we're using that } \Theta^0_{\alpha\beta}h^{\alpha\beta} = 0 \text{ and symmetric}\right)$$

(4.1) Complex algebraic geometry

To relate to something more familiar (possibly), we note that in local complex coordinates on Σ, a $\Theta \in \mathrm{Sym}^2_0(T^\vee\Sigma)$ has non-vanishing components Θ_{zz} and $\Theta_{\bar z\bar z}$ and on account of $\Gamma^z_{\bar z z} = \dots h_{\cdot\cdot} = 0$, the eq. $\nabla^\alpha\Theta_{\alpha\beta} = 0$ is simply

$$\bar\partial\,\Theta_{zz} = 0 \qquad \partial\Theta_{\bar z\bar z} = 0.$$

which is indeed one of the classical characterizations of tangent space to moduli of complex structures as "holomorphic quadratic differentials", similarly to CKG = "global holomorphic vector fields".

For another perspective, the (local, infinitesimal) change of coordinates that brings the deformed metric

$$\begin{pmatrix} \Theta_{zz} & h_{z\bar z} \\ h_{\bar z z} & \Theta_{\bar z\bar z} \end{pmatrix}$$

"back into diagonal form" is (not holomorphic, but rather) defined by

$$\frac{dz}{dz'} = 1 \qquad\qquad \frac{dz}{d\bar z'} = \frac{1}{2} h^{z\bar z}\, \Theta_{\bar z\bar z} = \mu^{z}_{\ \bar z}$$

s.t.

$$\begin{pmatrix} 1 & -\mu^{\bar z}_{\ z} \\ -\mu^{z}_{\ \bar z} & 1 \end{pmatrix} \begin{pmatrix} \Theta_{zz} & h_{z\bar z} \\ h_{\bar z z} & \Theta_{\bar z\bar z} \end{pmatrix} \begin{pmatrix} 1 & -\mu^{z}_{\ \bar z} \\ -\mu^{\bar z}_{\ z} & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(\Theta^2) & h_{z\bar z} \\ h_{z\bar z} & \mathcal{O}(\Theta^2) \end{pmatrix}$$

- Locally, this is always possible to do (i.e. $\mu^{z}_{\ \bar z} = \partial_{\bar z}\zeta^{z}$ for some (non-holomorphic) ζ^{z}, but globally on Σ $\mu^{z}_{\ \bar z}\, d\bar z$, known in this context as "Beltrami differential", will define, modulo global non-holomorphic vector fields, a non-trivial cohomology class

$$\mu \in H^1(T\Sigma). \qquad \text{(Čech or Dolbeault coh.)}$$

- Again this is akin to $\{CKV\} = H^0(T\Sigma)$, in which context the statement

[illegible]

$$\dim_{\mathbb{C}}[CKV] - \dim_{\mathbb{C}} \mathcal{M}_g = \dim \operatorname{Ker} P - \dim \operatorname{Ker} P^{+}$$
$$= \dim H^0(T\Sigma) - \dim H^1(T\Sigma)$$
$$= 3 - 3g$$

is a consequence of Riemann-Roch theorem, which states in general

$$\dim H^0(\Sigma, \mathcal{L}) - \dim H^1(\Sigma, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

for line bundle $\mathcal{L}$ on surface Σ of degree $\deg \mathcal{L}$.

(for $\mathcal{L} = T\Sigma$ of degree $\chi(\Sigma) = 2 - 2g$)

while

$$H^1(T\Sigma) = H^0((T^{\vee}\Sigma)^2)^{\vee}$$

quadratic differentials

is a manifestation of Serre duality, stating in general

$$H^0(\mathcal{L}) = H^1(\mathcal{L}^{\vee} \otimes K)^{\vee}$$

$T^{\vee}\Sigma$ = canonical bundle.

... returning to the real world...

At this stage, the "Lie map" $TG \to TM$ has the schematic form

$$\tilde{X} = \begin{array}{cc} & \begin{array}{cc} \mathcal{V}(\Sigma) = T\mathrm{Diff}(\Sigma) & T\mathrm{Weyl} \end{array} \\ \begin{array}{c} \mathrm{Sym}^2_0 \left\{ \begin{array}{c} TJ \\ TJ^{-1} \end{array} \right. \\ C^\infty(\Sigma)\, h_{\alpha\beta} \end{array} & \begin{pmatrix} 0 & 0 \\ P & 0 \\ \frac{1}{2}\nabla^\alpha \xi_\alpha\, h_{\alpha\beta} & 1 \cdot h_{\alpha\beta} \end{pmatrix} \end{array}$$

and the (Faddeev-Popov) determinant of the "effective" part is

$$\mathcal{D}_h = \left(\det{}' \tilde{X}^\dagger \tilde{X}\right)^{1/2} = \left(\det{}' P^\dagger P\right)^{1/2}$$

where $'$: ignore kernel / cokernel and trivial pieces.

· Namely

$$\mathcal{D}_h = \left(\det_{\mathcal{V}(\Sigma)/CKV} P^\dagger P\right)^{1/2}$$

$$= \det_{\mathrm{Hom}(\mathcal{V}(\Sigma)/CKV,\, TJ^{-1})} P$$

↑ calculated w.r.t. proper metrics.

· One can show

$$\text{“}\ \mathcal{D}_{e^{2\varphi}h} = \mathcal{D}_h\, e^{-26 S_L(\varphi, h)}\ \text{”}$$

(this is correct only after including the zero modes) so that the combined matter-metric determinant (calculated wrt non Weyl-invariant metric),

$$\frac{\mathcal{D}h}{\det(-\Delta_h)^{D/2}}$$

spacetime dimension, # of scalar fields, in general $\mathcal{L} + c^{\perp}$.

and thence all CFT correlation functions will be Weyl-invariant iff $D=26$, in which case we can integrate over $G = \mathrm{Diff}(\Sigma)/_{CKG} \ltimes \mathrm{Weyl}$ and remain with a finite-dimensional integral over $\mathcal{M} = \mathcal{T}/_{MCG}$ vs. a finite-dimensional residual global gauge group CKG (if $g=0,1$).

→ still need to do that!

Integral over $\mathcal{M}$ According to the discussion, the integral over $\mathcal{M}$ is to be done wrt. measure induced from Riemannian inner product on $T\mathrm{Met}(\Sigma)$.

In practice, this involves specification of local coordinates $(t^k)_{k=1,\dots \dim_{\mathbb{R}}(\mathcal{M})}$ and evaluation of

$$\mathcal{B} = \left[\det \langle \partial_\ell h, \partial_k h \rangle_{k,\ell}\right]^{1/2}$$

$$\parallel$$

$$\int_\Sigma d\mathrm{vol}_{\hat h}\, \hat h^{\alpha\gamma} \hat h^{\beta\delta}\, \partial_\ell h_{\alpha\beta}\, \partial_k h_{\gamma\delta}$$

on the chosen slice (e.g. $\hat h$ constant curvature metric)

· Alternatively, we can chose arbitrary (or orthonormal) basis $\Theta_{i\,\alpha}$ ($= \Theta_{i\,\alpha\beta}$, $\nabla^{\alpha}\Theta_{i\,\alpha\beta} = 0$) of $T\mathcal{T}$ and evaluate

$$\mathcal{B} = \frac{\det \langle \Theta_i, \partial_k h \rangle}{\det \langle \Theta_i, \Theta_j \rangle^{1/2}}$$

(The seeming potential non Weyl invariance of these expressions actually ensures previously stated transformation of D_h).

For this, one typically uses complex coordinates $(t^k)_{k=1,\dots \dim_{\mathbb{C}} \mathcal{M}}$ on $\mathcal{M}$ (and z on Σ) and replaces

$$\Theta_{i\,\alpha\beta} \longrightarrow \Theta_{i\,zz} \quad \text{(holomorphic quadratic differentials)}$$

$$\partial_k h \longrightarrow \mu_{k\,\bar{z}}^{\;z}\, d\bar{z}\, dt^k \quad \text{(Beltrami differentials)}$$

to write

$$\mathcal{B} = \Bigg| \underbrace{\det \int_\Sigma \Theta_{i\,zz}\, \mu_{k\bar{z}}^{\;z}\, dz\, d\bar{z}}_{=: (\Theta_i, \mu_k)} \Bigg|^2$$

which looks "manifestly Weyl-invariant".

<u>To deal with CKG</u>, we need to say a few things about vertex operator insertions.

- The basic idea for ensuring (conformal) invariance of matter correlation functions is to integrate weight (1,1) primary fields, which locally create OCQ physical states by operator-state correspondence, over Σ.

$$V_i = \int_\Sigma d^2 z_i \; V_i(z_i, \bar{z}_i)$$ (something similar for open strings)

and then insert $\prod_{i=1}^{n} V_i$ into matter path integral.

- ~~From~~ current perspective, diffeomorphism invariance is obvious, Weyl invariance is ensured after analyzing/defining vertex operators on curved worldsheets and writing in a general gauge

$$V_i = \int_\Sigma d^2\sigma \sqrt{\det \hat{h}} \; V_i(\sigma_i)$$ (with $V_i(\sigma_i)$ insertions)

- Alternatively, we can view this as an integration over an extension of our integration manifold to

$$\Sigma^n \times Met(\Sigma) \times Map(\Sigma, \mathbb{R}^{1,d})$$

wrt. to natural hyp ~~as~~ Σ itself on Σ.

- Physically, this can be thought of as integration over propagation times for intermediate strings (see GSW chapter 7 for extensive discussion)

- Mathematically,

$$\Sigma^n \times \mathrm{Met}(\Sigma) / \mathrm{Diff}(\Sigma) \ltimes \mathrm{Weyl} = \mathcal{M}_{g,n}$$

is "moduli space of Riemann surfaces with punctures", intuitively easy to picture but in practice somewhat difficult to describe/ parametrize because Diff also acts on Σs.

- Then, when $g \geq 2$, there are no CKV, the enlargement does not contribute to Lie map/ determinant and we recover previous formula.
- For $g=0,1$, there are dim CKG $= 3, 1$ copies of Σ that <u>are</u> part of gauge orbit. So we ~~can~~ have to fix that many points to pick the gauge slice at the price of an additional factor

$$C = \det \left(T\,CKG \longrightarrow \bigoplus_{i=1}^{\dim CKG} T\Sigma_i \, (z_i) \right)$$

fixed insertion point

Explicitly for $g=0$, $T\,CKG = l_{-1}, l_0, l_1$, the map is evaluation at the three points, s.t.

$$C = \det \begin{pmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{pmatrix} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$$

or rather $|\cdot|^2$ of this since we're using complex coordinates.

(where the orthonormalization of CKV belongs to $\mathcal{D}$, and there should also be factors of

$$\prod_{i=1}^{3} g_s \left(\det \hat{h}(z_i)\right)^{1/2}$$

for various reasons.

- This recovers fudge factor called ① in our evaluation of Veneziano/ Virasoro-Shapiro amplitude. It's the same for any number of insertions $n \geq 3$.

- For $g=0$, $n<3$, the amplitude can be declared 0 on account of either $\det P^{+}P = 0$ or infinite volume of CKG not being cancelled by integration of vertex operators.

- For $g=1$, special situation that CKG is compact and Σ is flat so that torus amplitude can be non-zero even for $n=0$ (or a "fictitious insertion" w/o V.O.). More later.

- Final result.
  - Geometrically pleasing and physically satisfying expressions for scattering amplitudes for ~~strings in~~ bosonic strings in $\mathcal{H}_{\text{string}} = \mathcal{H}_{OCQ}$ incarnation.

- More precisely, this requires arguments for decoupling of null states etc., which we won't give. See textbooks, or D'Hoker and Phong, RMP 60 No. 4, 917 (1988))

Instead: Explain how (I think) ghosts and BRST invariance fit into our geometric integration picture.