Lecture 13 etc. pp. 30.11.2021

③ Path-integral gymnastics

→ Namely, the (euclidean) path-integral with insertion of (local) operators gives a (formal) representation of (time-ordered) expectation value of corresponding quantum-mechanical observable (operator on Hilbert space representing that function restricted to phase space, in this context also known as quantum field, mathematically perhaps an operator-valued distribution), in the states given by boundary conditions.

Example: The two-point function of free boson on $\mathbb{C}$, with action

$$S = \frac{1}{2\pi\alpha'}\int d^2z \; \partial X \bar{\partial} X$$

defined (up to normalization) by

$$G = G(z,\bar{z};w,\bar{w}) = \langle X(z,\bar{z})\, X(w,\bar{w})\rangle = \int \mathcal{D}X\, e^{-S}\, X(z,\bar{z})\, X(w,\bar{w}).$$

is equal (according to standard rules of Gaussian integrals) to inverse of linear operator

$$-\frac{1}{2\pi\alpha'}\Delta = -\frac{1}{\pi\alpha'}\partial\bar{\partial} \qquad (\text{b/c } g^{z\bar{z}} = g^{\bar{z}z}).$$

conformally invariant operator!

therefore satisfies $\quad \partial\bar\partial G = -\pi\alpha' \delta(z-w)$
and is hence given by

$$G = -\frac{\alpha'}{2} \ln |z-w|^2$$

[the point being, $\displaystyle\int_{D_r(0)} f(z,\bar z)\, \bar\partial \frac{1}{z}\, d^2z = \int f(z,\bar z)\, \bar\partial \frac{1}{z} \left(-i\, d\bar z \wedge dz\right)$

$$dx\,dy = dx \wedge dy = -i\, d\bar z \wedge dz$$

$$= \underbrace{-i \oint_{C_r(0)} f(z,\bar z)\, \frac{1}{z}\, dz}_{= 2\pi f(0)} + \underbrace{i \int_{D_r(0)} \bar\partial f(z,\bar z)\, \frac{1}{z}\, d\bar z \wedge dz}_{=0 \text{ b/c } \bar\partial f \sim O(\bar z).}$$

(in $\lim_{r\to 0}$)

So $\quad \bar\partial \frac{1}{z} = 2\pi\, \delta^{(2)}(z).$]

and can equivalently be calculated from mode expansion in radial quantization

$$\partial X = -i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}$$

as follows: $\quad$ holomorphic

$$\langle 0|\, R\, \partial X(z)\, \partial X(w)\, |0\rangle \qquad |z|>|w|$$

$$= -\frac{\alpha'}{2} \sum z^{-n-1} w^{-m-1} \langle 0|\, \alpha_n \alpha_m\, |0\rangle$$

$$= n \qquad n = -m > 0$$

$$= 0 \qquad \text{otherwise}$$

$$= -\frac{\alpha'}{2} \frac{1}{z^2} \sum n \left(\frac{w}{z}\right)^{n-1} = -\frac{\alpha'}{2} \frac{1}{(w-z)^2}$$

& analogous for $|w|>|z|$; recover G by integration and $X = X_h + X_{\bar h}$.

Not that G (has no discontinuities other than at $z=w$ and) does not depend at all on quantization scheme or choice of origin in $\mathbb{C}$. In a rather precise sense, the Hilbert space is a local concept around each insertion points.

- For example, inserting $\partial X(0)$ in path integral looks in radial quantization like

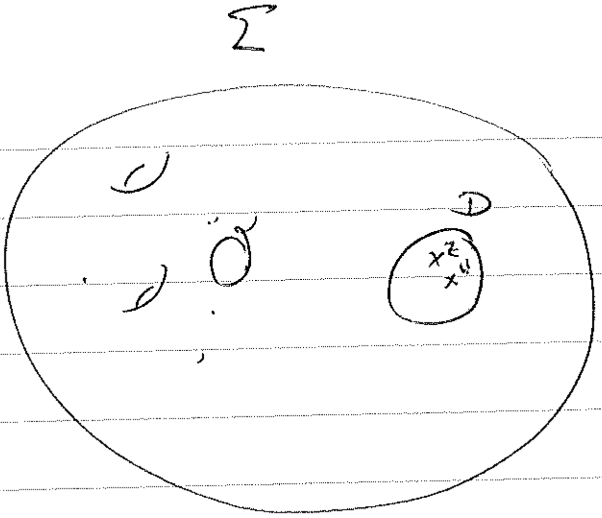
~~(depending no insertions between 0)~~

$$i\int_{\alpha'}^{2}\partial X(0)\rangle = \lim_{z\to 0}\sum \alpha_n z^{-n-1}|0\rangle .$$

$$= \lim_{z\to 0}\sum_{n<0}\alpha_n z^{-n-1}|0\rangle = \alpha_{-1}|0\rangle$$

like we have replaced the vacuum with $\alpha_{-1}|0\rangle$, the first excited state.

- {This (is no derivation, and also wrong at the end unless because we should have excluded $\alpha_{-1}|0\rangle$ from physical string spectrum (a zero momentum photon) even though it satisfies physical state condition) gives further intuition for scattering amplitude formula.
- Turning this around, the Hilbert space picture is useful to express / interpret the local behaviour of correlation functions when insertions approach each other.
- For example, the above analysis on $\mathbb{C}$ shows that $\partial X(z)\,\partial X(w)$ in correlation functions involving $\partial X(z)\partial X(w)$ on any 2-d surface Σ will always diverge as $\frac{1}{(z-w)^2}$ when $z\to w$.



$$= \int \mathcal{D}X \, e^{-S_\Sigma} \, \partial X(z) \partial X(w) \cdot \overset{\text{on } \Sigma \setminus D}{\mathcal{O}}$$

$$= \int \mathcal{D}Y \underbrace{\int_{X|_{\partial D} = Y} \mathcal{D}X_D \, e^{-S_D} \, \partial X(z) \partial X(w)}_{\text{" defines a state in Hilbertspace on } D \text{ "} \;\sim\; \frac{1}{(z-w)^2}} \cdot \int_{X|_{\partial D} = Y} \mathcal{D}X_{\Sigma\setminus D} \, e^{-S_{\Sigma\setminus D}} \, \mathcal{O}$$

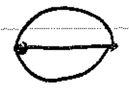
This (above considerations) precludes, among other things, the definition of vertex operators for states involving more than one oscillator such as $\alpha_{-1}\alpha_{-1}|0\rangle$ by the naive $\partial X(0)\,\partial X(0)$ (at best: ill-defined, at worst: infinite) but also suggest a way to repair:

$$:\partial X \partial X:(z) := \lim_{w \to z} \left[\partial X(z) \partial X(w) + \frac{\alpha'}{2} \frac{1}{(z-w)^2} \right]$$

is an (always) finite operator, corresponds in local Hamiltonian picture to normal ordering w.r.t. radial quantization (and is therefore addressed as such, even though the def does not depend on it!) and will therefore as a vertex operator as $z \to 0$, create $\alpha_{-1}\alpha_{-1}$ out of the vacuum.

(<u>AGAIN</u>: Not physical!)

· This need for normal ordering is familiar from perturbative treatment of interactions in QFT, usually in statement of no self-contractions in Wick theorem

$$:\phi(x)^3: \; :\phi(y)^3: \; = 6$$ 

· (Most likely) new in 2d conformal field theory is the extension to world surfaces of arbitrary topology and the combination for full power with the

④ Operator product expansion

General idea: · Local operators in QFT (fundamental fields = integration variables or other classical functions on field space rendered well-defined by some normal ordering, renormalization, etc. }; corresponding to quantum observables in canonical formalism) cannot naively be multiplied at coincident points because of short distance singularities.

· Instead, one (Wilson, Wilson-Zimmermann) assumes/ expects / postulates that if

$\{\mathcal{O}_i(0)\}_{i\in I}$ some countable set, e.g. polynomials in fund. fields and their multi-derivatives.

is "complete set of local operators", then $\exists$ functions w/ singularities $f_{ij}^{k}(\delta)$ s.t. "under path-integral" or "inside time-ordered correlation functions", rotational invariance

$$\mathcal{O}_i(\sigma)\,\mathcal{O}_j(\rho) = \sum_k f_{ij}^{\;k}(|\sigma-\rho|)\,\mathcal{O}_k(\rho)$$

and that this expansion converges weakly in small regions that do not contain other operators. Of course $f_{ij}^{\;k}$ are to be compatible with symmetries. In particular, in a conformal field theory, if $\mathcal{O}_i$ are operators with particular scaling dimension Δ_i (not nec. primaries) they transform as

$$\mathcal{O}_i(\sigma) = \lambda^{\Delta_i}\,\mathcal{O}_i(\lambda\sigma) \qquad \left(\rho = \lambda\sigma \quad \frac{d\rho}{d\sigma} = \lambda\right)$$

under "local rigid" rescaling by constant λ. Then (using rotational invariance):

$$f_{ij}^{\;k}(\lambda\delta) = \lambda^{\Delta_k-\Delta_i-\Delta_j}\,f_{ij}^{\;k}(\delta)$$

$$\Rightarrow \quad f_{ij}^{\;k}(\lambda\delta) \propto \frac{1}{|\delta|^{\Delta_i+\Delta_j-\Delta_k}}$$

Example: In 2d free boson (is chiral, hence holomorphic split)

"identity operator"

$$\partial X(z)\,\partial X(w) = -\frac{\alpha'}{2}\,\frac{1}{(z-w)^2} + :\partial X(z)\,\partial X(w):$$

because $(h,\hat{h}) = (1,0)$

because spacetime dimension, if you remember

$$= -\frac{\alpha'}{2}\frac{1}{(z-w)^2} + :\partial X^2(w): + (z-w) :\partial^2 X\, \partial X(w): + \dots$$

Laurent Taylor

— (usually only write principal part of Laurent expansion)

- In general interacting QFT, OPE is complicated (but still useful in some circumstances beyond perturbation theory), and borderline non-existent (there are modern mathematical versions).
- In general free field theory, OPE pretty much completely boils down to Wick theorem

hol. on cons.

oo eg

$$T(z) := -\frac{1}{\alpha'} : \partial X \partial X(z):$$

$$= 2\pi \cdot (zz\text{ - component of canonical energy-momentum tensor})$$

s.t.

$$L_n = \frac{1}{2\pi i} \oint z^{n+2}\, T(z)\, dz$$

namely

T is chiral of weight $(2,0)$

$$T = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$L_n = \frac{1}{2}\sum \alpha_p \alpha_{n-p}$$

Then, if necessary by a local consideration and appeal to canonical formalism,

$$T(z)\,\partial X(w) = -\frac{1}{\alpha'} :\partial X \partial X(z): \quad \partial X(w)$$

$$= -\frac{1}{\alpha'}\, 2 \quad \partial X(z) \cdot \Big(\text{contraction of } \underbrace{\partial X(z) \cdot \partial X(w)}_{= -\frac{\alpha'}{2}\frac{1}{(z-w)^2}} \Big)$$

$$+ \text{ regular}$$

$$= \frac{\partial X(z)}{(z-w)^2} + \text{regular}$$

$$= \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \text{regular}$$

- In general 2d CFT the OPE is so constrained ~~that~~ by symmetries that "theory can, at least in principle, be solved completely".
- We won't do much of this, but its ubiquitous power in string theory is reason enough that we discuss the beginnings here.

5 Ward identities

• To this end, recall from lecture 4: If $S = \int_\Sigma \mathcal{L}\, d^n\sigma$ and $\mathcal{L}$ is invariant under $\delta X = v$, then under $\delta X = \lambda(\sigma) v$
$\delta S' = \int_\Sigma j^\alpha \partial_\alpha \lambda \, d^n\sigma$ s.t. on eom

$$j_v^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\alpha X} v$$

closed $(n-1)$-form

is conserved $\partial_\alpha j^\alpha = 0$. Moreover $Q = \int_K \star j_v$ is charge generating symmetry in canonical formalism wrt. $\Sigma = I \times K$

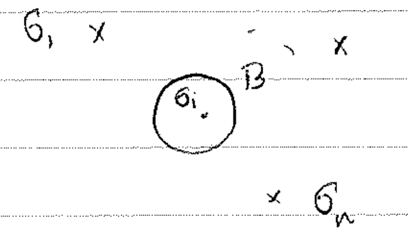
• In the path-integral formalism, we may view $\delta X = \lambda(\sigma) v$ as a mere "change of coordinates on field space" so that (akin to discussion of diffeomorphism invariance or simply $\int_M L_Y \omega = \int_{\partial M} \iota_Y \omega = 0$ if $\partial M = 0$ / M compact):

$$0 = \delta_{\lambda v} \int \mathcal{D}X\, e^{-S'}\, \mathcal{O}_1(\sigma_1) \cdots \mathcal{O}_n(\sigma_n)$$

$$= \int \mathcal{D}X\, e^{-S'} \Big(- \int_\Sigma j^\alpha \partial_\alpha \lambda \, d^n\sigma \; \mathcal{O}_1(\sigma_1) \cdots \mathcal{O}_n(\sigma_n)$$

$$+ \sum_i \mathcal{O}_1(\sigma_1) \cdots \delta_{\lambda v} \mathcal{O}_i(\sigma_i) \cdots \mathcal{O}_n(\sigma_n) \Big)$$

Choosing ~~λ to have support~~ $\lambda = \chi_B$ for B small nbd of σ_i



and using that $\mathcal{O}(\sigma_i)$ is local at σ_i, as well as Stokes theorem

$$\langle \mathcal{O}_1(\sigma_1) \ldots \delta_V \mathcal{O}_i(\sigma_i) \ldots \mathcal{O}_n(\sigma_n) \rangle = - \langle \int_{\partial B} * j_V^b \; \mathcal{O}_1(\sigma_1) \ldots \mathcal{O}_i(\sigma_i) \ldots \mathcal{O}_n(\sigma_n) \rangle$$

which is usually written as "operator Ward identity"

$$\delta_V \mathcal{O}(\sigma) = - \int_{\partial B} * j_V^b \; \mathcal{O}(\sigma)$$

In 2d CFTs.

- symmetries are chiral, meaning that any conserved current splits into

  $j^{\bar z}$ holomorphic

  j^{z} anti-holomorphic

  which are separately conserved (Lüscher-Mack theorem)

- $*dz = i dz$ $\quad *d\bar z = -i d\bar z$ so that on redefining $J = 2\pi \cdot$ (canonical j), one finds

$$\delta \mathcal{O}(w, \bar w) = \frac{1}{2\pi i} \oint_{C_\varepsilon(w)} \left(J_z dz - J_{\bar z} d\bar z \right) \mathcal{O}(w, \bar w)$$

which can be interpreted as the "residue" in the OPE of $J_z / J_{\bar z}$ with $\mathcal{O}$.

$$\delta \mathcal{O}(w, \bar w) = \mathrm{Res}_{z=w} J^z \mathcal{O}(w, \bar w) + \mathrm{Res}_{\bar z = \bar w} J^{\bar z} \mathcal{O}(w, \bar w)$$

or the action of the (quantum) observable/conserved charge
$$J_0 = \frac{1}{2\pi i}\int J_z\, dz$$

(the residue in the mode expansion
$$J_z = \sum_{n\in\mathbb{Z}} J_n (z-w)^{-n-1})$$

on the state created by $O(w,\bar{w})$ in a radial quantization around w.

Example: The current associated with translation $\delta X = 1$ of our free boson (featuring as spacetime coordinate of string) is

$$J_z = \frac{1}{\alpha'}\partial X = -i\sqrt{\frac{1}{2\alpha'}}\sum \alpha_n z^{-n-1}$$

$$\left(\text{i.e. } J_n = \frac{-i\,\alpha_n}{\sqrt{2\alpha'}}\right.$$

$$J_{\bar{z}} = \frac{1}{\alpha'}\bar{\partial} X = -i\sqrt{\frac{1}{2\alpha'}}\sum \tilde{\alpha}_n \bar{z}^{-n-1}$$

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}}\, p$$

$$p = \sqrt{\frac{2}{\alpha'}}\,\alpha_0 = 2i\,\mathrm{Res}\left(\frac{1}{\alpha'}\partial X\right) = 2\,\frac{1}{2\pi\alpha'}\oint \partial X\, dz$$
$$= \frac{1}{2\pi\alpha'}\oint \partial X\, dz.$$

- Therefore, to create a state with spacetime momentum p, we're looking for a (local) vertex operator $V_p(w,\bar{w})$ with residue

$$\alpha' J_0 = -\frac{i\alpha'}{2} p = \alpha' \bar{J}_0$$

in the OPE with ∂X, $\bar{\partial} X$.

- Natural candidate:

$$V_p(w,\bar{w}) = :e^{ipX(w,\bar{w})}:$$

indeed, by Wick's theorem and $\partial X(z)\, X(w,\bar{w}) = -\frac{\alpha'}{2}\frac{1}{z-w} + \text{regular}$

we have

$$\partial X(z)\, V_p(w,\bar{w}) = -\frac{i\alpha'}{2}\frac{1}{z-w} :e^{ipX(w,\bar{w})}: + \cdots$$

To be able to ensure that $V_p(w,\bar{w})$ creates a "spacetime physical state" (which we are claiming) we need to study the behaviour under conformal transformations

(6) Stress & energy

(where holomorphy of $T^{\bar{z}}{}_{z}$ "T_{zz}" / anti-holomorphy of $T^{z}{}_{\bar{z}}$ "$T_{\bar{z}\bar{z}}$" follows from its conservation and tracelessness
$T^{\bar{z}}{}_{\bar{z}} + T^{z}{}_{z} = 2T_{z\bar{z}} = 0$.)

T exists in any CFT, $= -\frac{1}{\alpha'}:\partial X \partial X:$ for free boson.

We claim:

First A local operator $\mathcal{O}(w,\bar{w})$ is a primary field

$$\delta_\zeta \mathcal{O}(w,\bar{w}) = h\,\partial\zeta\,\mathcal{O}(w,\bar{w}) + \zeta\,\partial\mathcal{O}(w,\bar{w}) + \text{a.h.}$$

if (and only if) the OPE with stress energy has principal part

$$T(z)\,\mathcal{O}(w,\bar{w}) = \frac{h\,\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \text{regular} \quad *$$

$$+\ \text{a.h.}$$

Pf: According lecture 4, the current associated with local conformal transformation generated by $\zeta\partial_z$, $\bar{\partial}\zeta=0$ is

$$J_\zeta = \zeta^z T_{zz} = \zeta T.$$

Then according to formula from complex analysis

$$\mathrm{Res}_{z=w} \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!}\,\partial^{(n-1)} f(w).$$

we have

$$\mathrm{Res}_{z=w} J_\zeta \mathcal{O}(w,\bar{w}) = h\,\partial\zeta(w)\,\mathcal{O}(w,\bar{w}) + \zeta\,\partial\mathcal{O}(w,\bar{w})$$

if $*$. If there were higher order poles, we'd have higher derivatives of ζ (possible). This happens for non primary fields.

<u>Second</u>: The state created by a primary field $\mathcal{O}(0,0)$ is "OCQ-physical" (i.e. $L_n - \delta_{n,0} = 0$, $n \geq 0$) iff $h = \tilde{h} = 1$ (a.k.a highest weight states)

<u>Pf</u>: L_n is charge ~~corrspon~~ associated with symmetry generated by $\zeta = z^{n+1}$. Then ("under path-integral", in correlators)

$$L_n \, \mathcal{O}(0,0) = \frac{1}{2\pi i} \oint z^{n+1} T(z) \, \mathcal{O}(0,0)$$

$$= \mathrm{Res}_{z=0} \, z^{n+1} \left(\frac{h \mathcal{O}}{z^2} + \frac{\partial \mathcal{O}}{z} + \dots \right)$$

$$\begin{cases} = h\mathcal{O} & n=0 \\ = 0 & n>0 \end{cases}$$

$$= \partial \mathcal{O} \quad n=-1 \quad \text{(translation)}.$$

<u>Third</u> $V_p(w,\bar{w}) = :e^{ipX(w,\bar{w})}:$ is primary of weight $h = \tilde{h} = \frac{\alpha' p^2}{4}$

<u>Pf</u>: $$T(z) V_p(w,\bar{w}) = -\frac{1}{\alpha'} :\partial X \partial X(z): \; :e^{ipX(w,\bar{w})}:$$

$$= -\frac{1}{\alpha'} (ip)^2 \left(-\frac{\alpha'}{2}\right)^2 \frac{1}{(z-w)^2} :e^{ipX(w,\bar{w})}:$$

$$- \frac{2}{\alpha'} :\partial X(z) \, e^{ipX(w,\bar{w})}: \; \frac{-ip\alpha'}{2(z-w)}$$

$$\partial X(z) X(w) = -\frac{\alpha'}{2} \frac{1}{z-w} + \dots$$

$$= \frac{\alpha' p^2}{4} \frac{:e^{ipX(w,\bar{w})}:}{(z-w)^2} + \frac{:\partial_w e^{ipX(w,\bar{w})}:}{z-w} + \text{regular}$$

(7) Outlook: • In OCQ vertex operators for physical states arise from primary fields of weight (1,1) in CFT w/ Hilbert space

$$\oplus_p \mathcal{F}^\perp_{p^\perp} \otimes \mathcal{H}^\perp$$

where $\mathcal{H}^\perp$ is unitary Virasoro module of central charge $c^\perp = 24$. E.g. when $\mathcal{H}^\perp = \oplus \bigotimes_{I=2}^{d} \mathcal{F}_{p^I}$,

• $:e^{ip_\mu X^\mu}:$ $\quad p^2 = -m^2 = \frac{4}{\alpha'}$ is vertex operator for tachyon at momentum p.

• $\varepsilon_{\mu\nu} :\partial X^\mu \bar{\partial} X^\nu e^{ip_\mu X^\mu}:$ $\quad p^2 = 0$ creates massless state of polarization $\varepsilon_{\mu\nu}$.

primality $\rightsquigarrow$ $p^\mu \varepsilon_{\mu\nu} = 0$ / $\varepsilon_{\mu\nu} = p_\mu a_\nu$ ← decoupling

$\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$ graviton

$\varepsilon_{\mu\nu} = \eta_{\mu\nu}$ dilaton

$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$ antisymmetric 2-form "B-field".

etc.

Complete and harmonious formulas for scattering amplitudes (especially at higher genus) require inclusion of ghosts also on the string worldsheet → Next lecture. do Veneziano, after that general story.

8 Complements

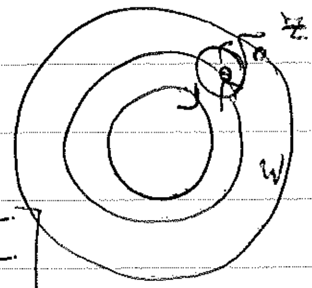
· In a general CFT, the stress-energy tensor T itself is not a primary field (of putative weight $(2,0)$). The OPE with itself is of the form

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular}.$$

This can be understood, for example, from relation to Virasoro algebra satisfied by its modes

$$L_n = \frac{1}{2\pi i}\oint z^{n+1}\,T(z)\,dz \qquad T(z) = \sum L_n z^{-n-2}.$$

$$\therefore \quad [L_n, L_m] = \oint\oint T(z)\,z^{n+1}\,T(w)\,w^{m+1}$$



$$= \oint dw\, w^{m+1} \oint_{C_\varepsilon(w)} dz\, z^{n+1}\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T}{z-w}\right]$$

$$= \int dw\, w^{m+1}\left[\frac{c}{12}(n+1)n(n-1)\,w^{n-2} + 2(n+1)\,w^{n}\,T(w) + w^{n+1}\,\partial T(w)\right]$$

$$= \oint dw \left(\frac{c}{12}(n^3-n)\, w^{n+m-1} + 2(n+1)\, w^{n+m+1}\, T(w) - (n+m+2)\, w^{n+m+1}\, T(w) \right)$$

$$= (n-m)\, L_{n+m} + \frac{c}{12}(n^3-n)\, \delta_{n+m,0}$$

(to go back, write $T(z) = \sum L_n z^{-n-2}$ and normal order)

This ~~kind of~~ violation of primarity for stress-energy tensor has a number of interesting consequences.

- First of all, for a general infinitesimal conformal transformation generated by ζ, we have

$$\delta_\zeta T(w) = \operatorname{Res}_{z=w} \zeta T(z)\, T(w)$$

$$= \underbrace{2\,\partial\zeta\, T(w) + \zeta\, \partial T(w)}_{\text{classical behavior}} + \frac{c}{12} \underbrace{\partial^3 \zeta}_{=0 \text{ for } \zeta = 1, z, z^2\ (l_{-1}, l_0, l_1)}$$

$\rightarrow$ quasi-primary

- This integrates for finite (but local) conformal transformations $z = F(w)$.

$$F^* T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \{z, w\}$$

where $\{z,w\} = \left(\frac{z''}{z'}\right)' - \frac{1}{2}\left(\frac{z''}{z'}\right)^2$

is the Schwarzian derivative (see exercises).

- For example, the transformation $z = e^{-iw}$ mapping the cylinder to the plane has

$$\{z, w\} = \frac{1}{z}$$

which is enshrined in the statement that "the" Hamiltonian governing time evolution on the cylinder is related to $L_0, \bar{L}_0$ (zero-modes of normal-ordered energy-momentum tensor in radial quantization) as

$$H_{cylinder} = L_0 + \bar{L}_0 \underbrace{- \frac{c + \tilde{c}}{24}}_{\text{"Casimir energy."}}$$

- By a slightly convoluted argument, one can also derive from the TT OPE the statement known as Weyl (or trace) anomaly: While conformal invariance of the action implies that, classically, the energy-momentum is trace-less (see lecture 3),

$$T^{\alpha}{}_{\alpha} = h^{\alpha\beta} T_{\alpha\beta} = 0$$

quantum-mechanically, it aquires a non-zero (vacuum) expectation value on curved backgrounds

$$\langle T^{\alpha}{}_{\alpha} \rangle = -\frac{c}{12} R$$

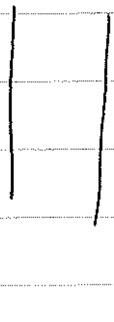
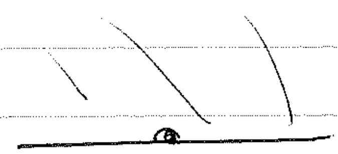
Ricci scalar.

- In free field theory, the Weyl anomaly can be understood from transformation of (ζ-function regularized) determinant of kinetic operator $\partial\bar{\partial}$ under Weyl rescaling.
- In general, recall that (classical) vanishing of $T_{z\bar{z}}$ was used ~~to derive~~ (with conservation of energy-momentum) to derive holomorphicity of T_{zz}. This however clashes with TT OPE

$$TT \sim \frac{c/2}{(z-w)^4} + \dots \quad , \quad \bar{\partial}\, \frac{1}{(z-w)^4} \text{ is not quite } 0.$$

- For detailed discussion of this and many other goodies, see lecture notes of D. Tong or big book by di Francesco et al.

Open strings: Were more convenient to discuss ~~the~~ Hilbert space, but require additional considerations in path-integral (boundary conditions) or operator formalism (boundary conformal field theory). In the end, it's not that complicated.

Fact:
The strip $\mathbb{R}\times[0,\pi]$  is conformally equivalent to 

upper half plane $\mathbb{H}$
$= \{\operatorname{Im} z \geq 0\} \subset \mathbb{C}$

- open strings are created by vertex operators inserted at the boundary (and interact with closed strings inserted in the interior).

- (local) conformal transformations of $\mathbb{H}$ are generated by holomorphic vector fields ζ that leave the boundary invariant, i.e.

$z = x + iy.$

$$\zeta^y = \frac{1}{2i}\left(\zeta^z - \zeta^{\bar z}\right) = 0 \quad \text{at } z = \bar z$$

$\zeta_n^z = z^{n+1}$ $\zeta_n^{\bar z} = \bar z^{n+1}$ satisfy this condition (but may now only be multiplied by real numbers).

Global: l_{-1}, l_0, l_1 generated $SL(2,\mathbb{R})$, real fractional linear transformations.

- local operators inserted at the boundary are ~~pri~~ (boundary) primary if

$$\delta\mathcal{O}(x) = h\,\partial_x\zeta\,\mathcal{O} + \zeta\,\partial_x\mathcal{O}.$$

- physical open string states are created by boundary primary fields of weight 1.

- Example: Open string tachyon vertex operator at momentum p.

$$V_p(x) = :e^{ip\cdot X(x)}:$$

$$h = \alpha' p^2 = -\alpha' m^2 = 1.$$

To verify this, use bulk-boundary and boundary-boundary OPEs derived from radial quantization of open string using "doubling trick" / method of images.

$$\partial X = -i\sqrt{\frac{\alpha'}{2}}\sum \alpha_n z^{-n-1} \qquad \bar\partial X = -i\sqrt{\frac{\alpha'}{2}}\sum \alpha_n \bar z^{-n-1}$$

$$(\alpha_n = \tilde\alpha_n \ !)$$

satisfying $\partial X = \bar\partial X$, i.e. $\partial_y X = 0$ at $z = \bar z$

(Neumann boundary condition)

For $z, w \in H$ we still have $\partial X(z)\, \partial X(w) = -\frac{\alpha'}{2}\frac{1}{(z-w)^2}$

but (near boundary!) $\partial X\, \bar\partial X = -\frac{\alpha'}{2(z-\bar w)^2}$

for a total $\qquad \partial_x = \partial_z + \partial_{\bar z} = d_x$

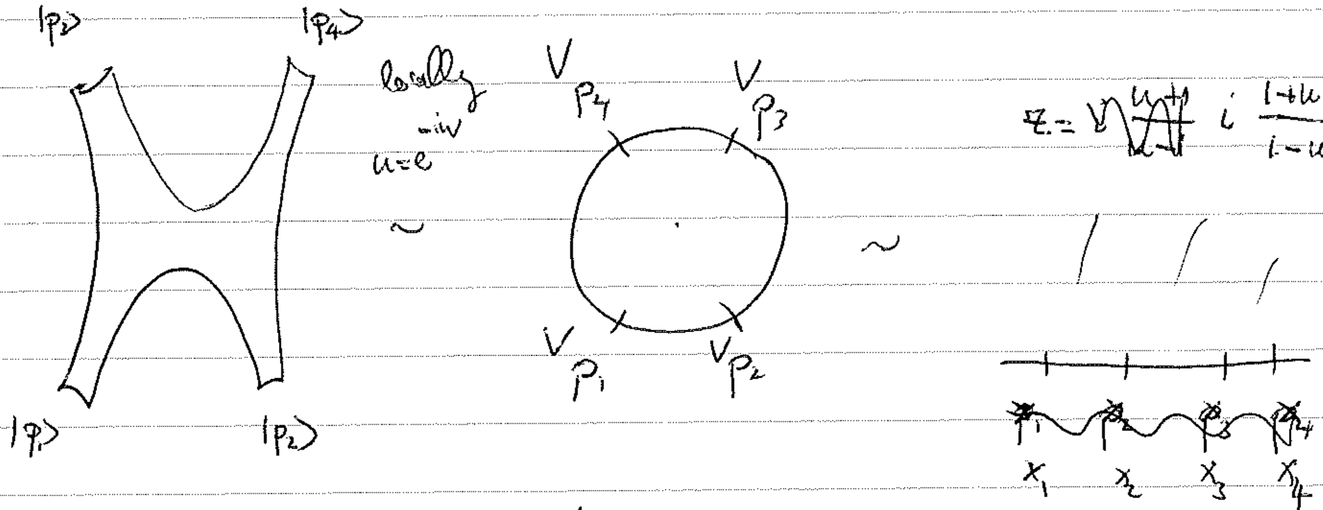
$$\partial X(z)\, \partial_x X(x) = -\alpha' \frac{1}{(z-x)^2}$$

$$d_x X(x)\, d_x X(y) = -\frac{2\alpha'}{(x-y)^2}$$

$$p = i \operatorname{Res} \frac{\partial X}{\alpha'} = \frac{1}{2}\, i \operatorname{Res} \frac{\partial X + \bar\partial X}{\alpha'} \quad \text{etc.}$$

Lecture 14. Venetiano amplitude

- String theory was born when the amplitude proposed in 1968 by G. Veneziano to account for proliferation of hadronic resonances of high spin in (fairly low energy) pion-nucleon ~~scattering~~ collision was understood (by ??? Nielsen, Susskind...) to describe 2-to-2 scattering of open string tachyons (see Green-Schwarz-Witten for beautiful account of history).
- We can reproduce this simplest non-trivial result by evaluating the diagram



starting from the formula

$$A = A(p_1,p_2,p_3,p_4) = \int \frac{\mathcal{D}h\,\mathcal{D}X}{\mathrm{Diff}\ltimes\mathrm{Weyl}}\, e^{-S_p(X,h)} \prod_{i=1}^{4} V_{p_i}(x_i)$$

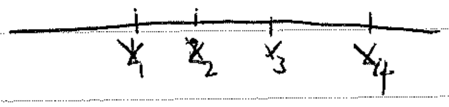
(and a bit of hand-waving; more details next time).
justification

<u>Fact:</u> · Any metric on the upper half-plane is (not just locally but even) globally conformally flat, with room to spare:

· The group of global conformal transformations of $\mathbb{H}$ is isomorphic to $PSL(2,\mathbb{R})$ acting by fractional linear transformations and allows us to fix three of the four insertion points, say x_1, x_3, x_4, arbitrarily, with the fourth one x_2 remaining as single parameter (real) of the metric h not fixed by diff$\ltimes$Weyl.

· (Actually, we should sum over some number of cyclic orderings, but forget this).

· Namely



$$A = \int_{x_1}^{x_3} dx_2 \left\langle V_{p_1}(x_1)\, V_{p_2}(x_2)\, V_{p_3}(x_3)\, V_{p_4}(x_4) \right\rangle_{\mathbb{H}} \quad \Phi$$

where Φ is a "fudge factor" to ensure that A does not depend on how we fixed x_1, x_3, x_4.

Under $x = \dfrac{ay+b}{cy+d}$ $(ad-bc=1)$ the V_p (being bdry conformal primaries of weight 1) transform as

$$V_{p_i}(y_i) = \frac{dx}{dy} V_{p_i}(x_i) \qquad \frac{dx}{dy} = \frac{1}{(cy+d)^2}$$

$$V_{p_i}(x_i) = (cy_i+d)^2\, V_{p_i}(y_i)$$

· This is compensated by

$$dx_2 = \frac{1}{(cy_2+d)^2}\, dy_2 \qquad \left(V_{q_2}(x_2)\, dx_2 \text{ is one-form} \right)$$

$$\mathcal{D} = |x_1 - x_3|\,|x_1 - x_4|\,|x_3 - x_4|$$

$$\left(\text{the pt. being} \quad x_i - x_j = \frac{y_i - y_j}{(cy_i+d)\,(cy_j+d)} \right)$$

· From $V_p(x) = \, :e^{ip_\mu \cdot X^\mu(x)}:$

$$X^\mu(x)\, X^\nu(y) = -2\alpha' \ln|x-y|\, \eta^{\mu\nu}$$

we find

$$\langle \prod V_{p_i}(x_i) \rangle_H \sim \text{contractions} \cdot \langle : \prod V_{p_i}(x_i) : \rangle$$

$$= \prod_{i<j} |x_i - x_j|^{2\alpha' p_i \cdot p_j} \cdot \underbrace{\langle : e^{i(p_1+p_2+p_3+p_4)\cdot X} : \rangle_{H}}_{=0 \text{ unless } p_1+p_2+p_3+p_4}$$

more precisely our derivation of Green's function did not account for zero mode of X, and

$$\int e^{i(p_1+p_2+p_3+p_4)\cdot x}\, d^D x = (2\pi)^D\, \delta^D(p_1+p_2+p_3+p_4)$$

which we will strip / embezzle in phase space factor that converts amplitude to differential cross-section

$$\frac{d\sigma}{d\Omega} = |A|^2 \cdot \text{phase space factor}.$$

Under the conventional choice $(x_1\ x_3\ x_4) = (0\ 1\ \infty)$

$$|x_1 - x_4|^{2\alpha' p_1 p_4} |x_2 - x_4|^{2\alpha' p_2 p_4} |x_3 - x_4|^{2\alpha' p_3 p_4} \quad |x_1 - x_4||x_3 - x_4|$$

$$= |x_4|^{2\alpha' p_4 \underbrace{(p_1+p_2+p_3)}_{=-p_4} + 2} = 1 \qquad p_4^2 = \frac{1}{\alpha'}$$

and so

$$A = \int_0^1 dx \quad x^{2\alpha' p_1 \cdot p_2} \quad |1-x|^{2\alpha' p_3 \cdot p_2}$$

This can be written in terms of Euler

$$\Gamma(u) = \int_0^\infty x^{u-1} e^{-x}\, dx \qquad u\Gamma(u) = \Gamma(u+1)$$

and Beta function

$$B(u,v) = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)} = \int_0^1 dx \quad x^{u-1} \, (1-x)^{v-1}$$

$$\Gamma(u)\Gamma(v) = \iint dx dy \quad x^{u-1} e^{-x} y^{v-1} e^{-y} = (x+y=z, \quad x \to zx)$$

$$= \int_0^\infty dz \quad z^{u+v-1} e^{-z} \int_0^1 dx\, x^{u-1} (1-x)^{v-1}. = \Gamma(u+v)\ B(u,v)$$

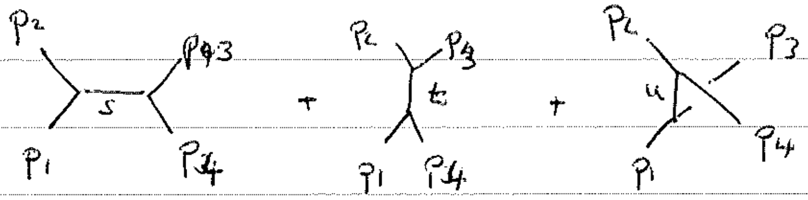
$$p_i^2 = \frac{1}{\alpha'} = -m^2$$

and Mandelstam variables for $2 \to 2$ scattering

$$s = -(p_1+p_2)^2 = -\frac{2}{\alpha'} - 2p_1p_2$$

$$t = -(p_2+p_3)^2 = -\frac{2}{\alpha'} - 2p_2p_3$$

$$u = -(p_1+p_3)^2 = -\frac{2}{\alpha'} - 2p_1p_3 = -\frac{4}{\alpha'} - s - t$$



$$A = B(-\alpha(s), -\alpha(t))$$

where $\alpha(s) = \alpha' s + \alpha(0)$ $\qquad \alpha(0) = 1$.

Regge slope $\qquad$ intercept.

<u>Interpretation</u>

$$\mathrm{Res}_{u=-n} \Gamma(u) = \frac{(-1)^n}{n!}$$

and some considerations at ∞ imply that for fixed v

$$B(u,v) = \sum_{n=0}^{\infty} \frac{(v-1)\cdots(v-n)}{n!} \frac{(-1)^n}{u+n}$$

converges $\forall\, u \in \mathbb{C}$ (Weierstrass) whence

$$A = -\sum \frac{(\alpha(s)+1)\,(\alpha(s)+2)\cdots(\alpha(s)+n)}{n!}\;\frac{1}{\alpha(t)-n}$$

can be viewed as arising from exchange in t-channel of particles of mass

$$m^2 = t = \frac{n-\alpha(0)}{\alpha'}$$

and spin J "up to n" [this follows from numerator arising from coupling of type

$$\phi^* \overleftrightarrow{\partial}_{\mu_1} \cdots \overleftrightarrow{\partial}_{\mu_J} \phi \;\sigma^{\mu_1 \cdots \mu_J}$$

totally symmetric

prop. of σ . $\dfrac{\eta^{\mu_1\nu_1}\cdots\eta^{\mu_J\nu_J}}{t-m^2}$

$$(p_3-p_2)(p_4-p_1) = p_3p_4 + p_1p_2 - p_3p_1 - p_2p_4 = t-s$$

$$s = p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4$$

$$t = p_2p_1 + p_2p_4 + p_3p_1 + p_3p_4$$

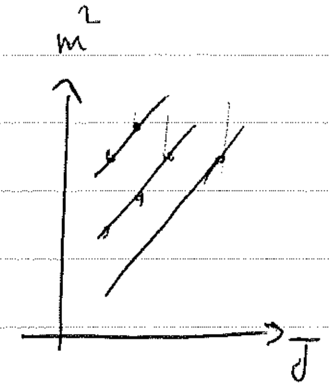
in the limit of large s, fixed t

(Regge limit).

or equivalently as exchange of exactly the same particles in s-channel (duality hypothesis)

· The dependence

$$J = \alpha' m^2 + \alpha(0)$$



had been studied in Regge theory (analytic continuation in angular momentum), matched experimental data and as we have see is characteristic of strings. (lecture 2).

$p_1 = (E, \vec{p})$ CM frame, $\vec{p}^2 = E^2$ (massless particles)

$p_2 = (E, -\vec{p})$

$p_3 = (E, \vec{p}')$

$p_4 = (E, -\vec{p}')$

$s = 4E^2 \qquad t = 2E^2 - 2E^2 \cos\theta = -\frac{s}{2}(1 - \cos\theta)$.

· Regge limit $s \to \infty$, fixed t : good agreement.

· hard scattering at fixed angle ($\frac{s}{t}$ fixed) : exponential fall off, in reality polynomial behaviour from parton constituents $\longrightarrow$ QCD.

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