Lecture 11 String interactions (A SKETCH) 23.11.21

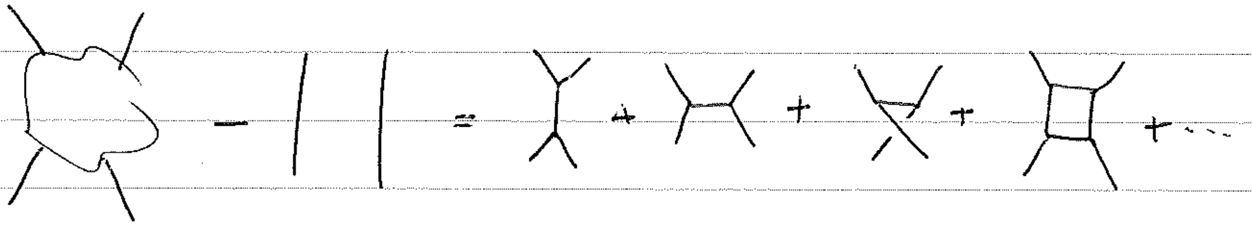
So far: Construction of $\mathcal{H}_{string}$ by quantization of dynamical system defined by Nambu-Goto / Polyakov action on $\mathbb{R}\times S^1$ (or $\mathbb{R}\times I$ for open strings).

Next goal: Give prescription for calculation of transition amplitudes that account for interactions between several strings.

① Intuition from field theory:
- Full Hilbert space is Fock space built on $\mathcal{H}_{particle}$, represents canonical quantization of action for spacetime fields $\phi(x)$ that "create particles at spacetime points" $x \in \mathbb{R}^{1,d}$.
- interactions are introduced by deforming quadratic/free action (while respecting symmetries).
- transition amplitudes ~~can~~ arise by certain reduction procedure from correlation functions

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle$$

of interacting fields that can (for example) be formally defined by Feynman path-integral but (most importantly) are calculable in perturbation theory by evaluating Feynman diagrams, e.g.



Slightly less well-known (though Feynman's original intuition)

This can also be obtained "directly" from particle picture, by evaluating one-dimensional path-integrals.

Example (Sketch) In (above) theory defined by free action

$$I = \frac{1}{2} \int \left(-\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) d^D \phi$$

the 2-pt function (propagator)

$$G(x,y) = \langle \phi(x)\, \phi(y) \rangle = \frac{\int \mathcal{D}\Phi\ \phi(x)\, \phi(y)\, e^{iI(\phi(-))}}{\int \mathcal{D}\phi\ e^{iI(\phi(-))}}$$

$$= \cdots = \int d^D p\ e^{ip(x-y)} \frac{1}{p^2 + m^2}$$

(with main feature

$$\left(-\Box_x + m^2\right) G(x,y) = \delta^D(x-y)$$)

can be written as

$$= \int d^D p\ e^{ip(x-y)} \int_0^\infty dt\, e^{-t(p^2+m^2)}$$

$$= \int_0^\infty dt\ \langle y |\, e^{-t \cdot H}\, | x \rangle$$

$$= \int_0^\infty dt \int_{\substack{z(0)=x \\ z(t)=y}} e^{-S(z(-),t)}\ \mathcal{D}z(\cdot)$$

possibly off by factor of 2!

where
$$S'(z(\cdot)) = \frac{1}{2}\int_0^t (\dot{z}^2 - m^2)\, d\tau$$

NB: Euclidean P.I.

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{t}\dot{z}^2 - t m^2\right) d\tau$$

Namely, $\hat{G}(x,y) = \underset{\text{length of interval}}{\int dt} \underset{\text{paths } x\to y}{\int \mathcal{D}z(\cdot)}\, e^{-S(z(\cdot),t)}$

which is gauge-fixed version of

$$= \underset{\text{metrics on interval}}{\int de(\cdot)} \underset{\text{paths } x\to y}{\int \mathcal{D}z(\cdot)}\, e^{-S(z,e)} \Big/ \text{Diffeomorphisms}$$

Meanwhile, after deforming action by $\delta I \sim \lambda \int \phi^3 d^Dx$ Feynman amplitudes for trivalent graph Γ can be calculated equivalently as

$$\underset{\text{external legs}}{A(x_1,\dots x_n)} \sim \underset{\text{metrics } g \text{ on } \Gamma / \text{diffeos}}{\int} \lambda^{\#\text{vertices}} \underset{\substack{z:\Gamma\to\mathbb{R}^{1,d} \\ \text{w/ b.c.}}}{\int} e^{-S(z,g)}$$

details pending

<u>Remarks:</u> · This really defines "off-shell amplitudes", actual scattering requires LSZ reduction (and explicit evaluation renormalization etc.) ⊛

- This becomes cumbersome for particles with spin/gauge invariance. (I know only one source: Matt Strassler's PhD thesis), certainly
- no help for gravity.
- Allows more or less arbitrary deformations by coupling to background metrics (Feynman-Kac formula), where it provides useful insights into QFT in curved space (1-d theory becomes "interacting", no dynamical gravity in spacetime).
- interactions by hand.

⊛ e.g. for free propagation

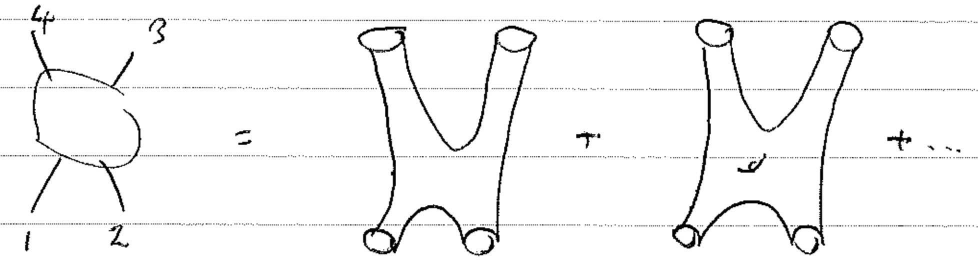
$$\langle p | q \rangle = \int e^{-ipx} e^{iqy} \Delta(x,y) \sim \frac{\delta(p-q)}{p^2+m^2}.$$

$\Longleftrightarrow$ momentum conservation.

but $p^2 = -m^2$ not necessary.

② Situation in String Theory

- As of 2021, we (still) do not (really) know how to (fully) define interacting strings from spacetime pt of view. Although e.g. string field theory does accomplish this as far as perturbation theory is concerned, it is (technically complicated, but still) only a reformulation of worldsheet formalism.
- Instead, we postulate/define scattering amplitudes to be given by the "natural generalization" of the worldline formalism, based on the 2-d Polyakov action, as a sum over string diagrams



with each worldsheet topology contributing

$$\int \prod_{i=1}^{4} \psi_i(X_i(-))\, \mathcal{D}X_i(-) \times$$

← Wave functionals of asymptotic states

$$\times \int \frac{\mathcal{D}h\, \mathcal{D}X}{\text{Diffeomorphisms} \times \text{Weyl}} e^{-S_P(X(-),h)}$$

b.c.: $X(\tau \to \tau_i, \sigma) = X_i(\sigma)$.

where

$$S_P(X(\cdot),h) = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \sqrt{h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$$

is the Polyakov action,

Remarks: · albeit in Euclidean signature – This is like for particles, more later. Note that we do not require analytic continuation in spacetime momenta although discussion of $i\epsilon$'s is interesting in string theory too.

· Also note that amplitude is only defined for on-shell external states – There are reasons not to worry and rather declare it a feature; string theory includes gravity and "gravity has no local observables" (spacetime diffeomorphism invariant local functional of fields).

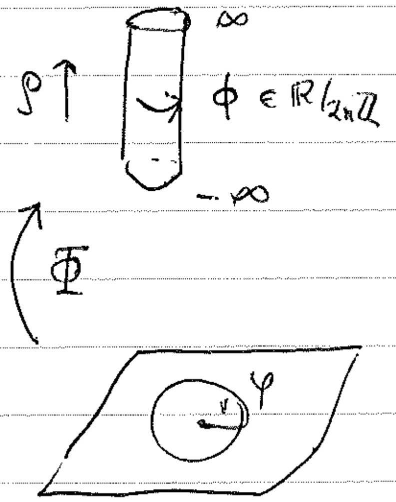
· Also from the worldsheet point of view, the restriction to on-shell external states is a consequence of insistance on diffeomorphisms and Weyl invariance as local (gauge) symmetries.

To relate more concretely to previous discussion (and anticipate the future), we note that ~~the~~ after fixing conformal gauge, while Σ will not have many global symmetries, "locally on each tube" there are ∞ many CKV generating globals symmetries that we gauge. This requires external states to be conformally invariant ($\hat{=}$ on-shell).

Imagine: External legs are $1\, l_Z \sim 10^{16}$ cm, interaction region $1\, l_P \sim 10^{-33}$ cm.

Diffeomorphism / residual conformal invariance (necessary to cancel negative norm states upon gauging) is also the basis for another central feature of string theory, the state-operator correspondence.

Fact: The 2-dimensional cylinder is conformally equivalent to the punctured plane.



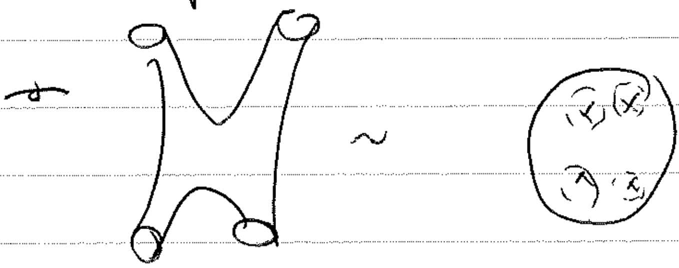
$$ds_C^2 = d\rho^2 + d\phi^2$$

$$ds_P^2 = dr^2 + r^2 d\varphi^2$$

$$\Phi(r, \varphi) = (\ln r, \varphi) = (\rho, \phi)$$

$$\Phi^*(ds_C^2) = \frac{dr^2}{r^2} + d\varphi^2 = \frac{1}{r^2}(ds_P^2).$$

$\rightsquigarrow$ 2d worldsheet with long tubes is (not only diffeomorphic, but even) conformally equivalent to compact surface with punctures



In the evaluation of the amplitude, the asymptotic state data is mapped to insertion of "vertex operator" (a local functional of matter fields, derivatives, metric, ghosts) in path integral.

$$\int \frac{\mathcal{D}h\,\mathcal{D}X}{\text{Diff} \ltimes \text{Weyl}} \; e^{-S_P(X,h)} \prod_{i=1}^{4} V_i(X,h)$$

State-operator correspondence: external states $\leftrightarrow$ local operators holds both before and after gauge fixing.

- Thus, in a remarkably precise sense, the "interactions are already present in the free theory".
- Correspondingly, there are no free parameters (analogous to λ) in string perturbation theory.
- Except that we need account for complexity of Σ.
  $\rightsquigarrow$ genus counting parameter g_s.
- Which however is also not really free, but arises as constant value of a target space field – the dilaton ϕ to which the string couples $\int \phi \cdot R^{(2)} = \phi\,\chi(\Sigma)$.
- More general couplings to spacetime fields also ~~allowed~~, possible but remarkably not all are allowed. More later.

Lecture 12 CFT, OPE, etc. 25.11.21

- Plan: Calculate string scattering amplitudes by "integrating correlation function of vertex operators corresponding to external states in a 2-d field theory, diff × Weyl invariant theory, for fixed metric over all metrics".
- After gauge fixing (details soon), : correlation function in 2-d conformal field theory on compact (euclidean) worldsheets
- Since we do not necessarily want to assume path-integral definition (of matter sector) or rely on it for actual calculations, we need to be somewhat flexible in switching back-and-forth with operator formalism (that can be compared with $\mathcal{H}_{string}$, among other things!)

Purpose: · Lightning overview of implications of conformal invariance in 2d field theory

· motivate / derive expression and properties of simple vertex operators.

Note: (2d) CFT is big subject with many applications and dedicated lectures, which we can impossibly summarize ~~also~~.

This section: mostly closed strings

1 Quick review

- a conformal Killing vector of (Σ, h) is $\xi \in \mathcal{X}(\Sigma)$ s.t.

$$\mathcal{L}_\xi h = \lambda h \qquad \left(\lambda = \frac{2}{n} \operatorname{div} \xi \right)$$

- For flat space $\mathbb{R}^{p,q}$, $n = p+q > 2$, metric $\eta_{\alpha\beta}$ diagonal, the Lie algebra of CKV is generated by

$$P_\alpha = \partial_\alpha \qquad \text{translations}$$

$$J_{\alpha\beta} = \sigma_\alpha \partial_\beta - \sigma_\beta \partial_\alpha \qquad \text{rotations}$$

Isometries

$$D = \sigma^\alpha \partial_\alpha \qquad \text{dilatation}$$

$$K_\alpha = \sigma^\beta \sigma_\beta \partial_\alpha - 2 \sigma_\alpha \sigma^\beta \partial_\beta \qquad \text{"special conformal generators"}$$

has dimension $n + \frac{n(n-1)}{2} + 1 + n = \frac{(n+2)(n+1)}{2}$, and is isomorphic to

$$so(p+1, q+1). \qquad \text{(check this!)}$$

Special role: D, eigenvalues known as conformal or scaling dimension.

$$[D, P_\alpha] = -P_\alpha \qquad [D, J_{\alpha\beta}] = 0 \qquad [D, K_\alpha] = K_\alpha$$

When $n=2$, in coordinates in which η is off-diagonal, the conformal Killing equation simplifies to

$$\partial_1 \xi^2 = 0 \qquad \partial_2 \xi^1 = 0$$

and the Lie-algebra of CKV is ∞-dimensional.
in Minkowski signature previously written as

$$\xi^+(\sigma^+)\partial_+ + \xi^-(\sigma^-)\partial_-$$

- In euclidean signature, we can make η off-diagonal (only) in complex coordinates $z, \bar{z}$ whence CKV are simply (real and imaginary parts of) holomorphic vector fields.

$$ds^2 = dz\,d\bar{z} \qquad g_{z\bar{z}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

- On $\mathbb{C}$, locally around $z=0$

$$l_n = z^{n+1}\partial_z \qquad [l_n, l_m] = (n-m)\, l_{n+m}$$

N.B.:

$$l_{-1} = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

$$l_0 = z\partial_z = \frac{1}{2}(r\partial_r - i\partial_\varphi)$$

$$l_1 = z^2\partial_z = \frac{1}{2}(x^2 - y^2 + 2ixy)(\partial_x - i\partial_y)$$

$l_0 + \bar{l}_0 = r\partial_r$ conformal dimension

$l_0 - \bar{l}_0 = -i\partial_\varphi$ conformal spin

l_0 : conformal weight.

are $sl(2,\mathbb{C})$ subalgebra, distinguished by the fact that they continue to $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. (Global conformal transformations.) ⊛

- Finite conformal transformations (continuously connected to identity) are invertible maps $\sigma^\pm = f^\pm(\tilde{\sigma}^\pm)$, or in euclidean signature, "local biholomorphisms".

$$F(z,\bar{z}) = \left(f(z), \bar{f}(\bar{z})\right)$$

under which metric

$$F^*(dz d\bar{z}) = |f'(z)|^2 dz d\bar{z}$$

(to be sure, inversions $\sigma^+ \leftrightarrow \sigma^-$ $z \leftrightarrow \bar{z}$ are also conformal).

⊛ $w = \frac{1}{z}$ $\quad \partial_z = -w^2\partial_w \quad \cancel{z^2} \; z^2\partial_z = -\partial_w \quad z\partial_z = -w\partial_w$

<u>NB:</u> Still call this conformal field theory, holomorphic has more restricted meaning.

② Conformal fields

The simplest (euclidean) field theory that has local conformal transformations as global symmetries is "free boson on a plane"

$$X : \underset{\underset{\mathbb{C}}{\parallel}}{\Sigma} \longrightarrow \mathbb{R}$$

with action that in euclidean signature and complex coordinates is given by

$$S_E = \frac{1}{2\pi\alpha'} \int_\Sigma d^2z \quad \partial_z X \, \partial_{\bar{z}} X$$

and which can be obtained from previously studied "scalar on cylinder"

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \, (\partial_\tau X)^2 - (\partial_\sigma X)^2$$

by

- "Wick rotation" $\quad \tau = -i\tilde{\tau}$

$$\rightsquigarrow \quad \tilde{S} = \frac{i}{4\pi\alpha'} \int d\tilde{\tau} d\sigma \, \left((\partial_{\tilde{\tau}} X)^2 + (\partial_\sigma X)^2 \right)$$

- followed by conformal map $\quad z = e^{i\sigma^-}$

$$\sigma^- = \tau - \sigma = -\underbrace{(\sigma + i\tilde{\tau})}_{\text{sometimes called } w}$$

NB: Factor of 2 from $g^{z\bar{z}} \partial_z X \partial_{\bar{z}} X + g^{\bar{z}z} \partial_{\bar{z}} X \partial_z X$. Map is conformal!!

$$F(\sigma^-) = z = e^{i\sigma^-} \rightsquigarrow \frac{d\sigma^-}{dz} = -\frac{i}{z}$$

$$F^*(d\sigma^+ d\sigma^-) = \frac{1}{|z|^2}\, dz\, d\bar{z}.$$

- multiplication by $-i$ s.t. $e^{iS'} = e^{-S'_E}$
- attention at $z=0$ corresponding via $z = e^{\tilde{\tau} - i\sigma}$ to $\tilde{\tau} \to -\infty$.

- S'_E (from now on renamed S') can be studied in canonical formalism (and quantized) wrt any choice of (euclidean) time.

Comparison with $\mathcal{H}_{\text{string}}$ (construction of vertex operators etc.) is possible in "radial quantization", constructed as conformal trafo of cylinder such that "time" evolution given by

$$D = l_0 + \bar{l}_0 = r\partial_r \sim \partial_{\tilde{\tau}}$$

(up to a constant) and external state at $r=0$.

⋮

Skipping details

⋮

⇝ Solutions of classical eoms

$$\frac{\delta S}{\delta X} = \partial_z \partial_{\bar z} X = 0$$

(which say that $X = X_L(z) + \tilde X(\bar z)$)

are to be parametrized as $z = e^{i\sigma^-}$ $\bar z = e^{i\sigma^+}$

$$X_L(z) = \frac{1}{2} x + \sqrt{\frac{\alpha'}{2}} \left(-i \ln z\right) \alpha_0 + i \sqrt{\frac{\alpha'}{2}} \sum \frac{\alpha_n}{n} z^{-n}$$

$$\tilde X_{\tilde L}(\bar z) = \frac{1}{2} x + \sqrt{\frac{\alpha'}{2}} \left(-i \ln \bar z\right) \tilde\alpha_0 + i \sqrt{\frac{\alpha'}{2}} \sum \frac{\tilde\alpha_n}{n} \bar z^{-n}$$

with coefficients satisfying reality conditions $\alpha_n^\dagger = \alpha_{-n}$ and Poisson brackets $\{\alpha_n, \alpha_m\} = i n \delta_{n,-m}$ $\underbrace{}_{\text{"reflection positivity"}}$ as before.

Although $\alpha_0 = \tilde\alpha_0$ implies $X_L + \tilde X_{\tilde L}$ is well-behaved, $\log z$ is slightly inconvenient. Better consider

$$\partial X = \partial X_L = -i \sqrt{\frac{\alpha'}{2}} \sum \alpha_n z^{-n-1} = \frac{\partial \sigma^-}{\partial z} \partial_{\sigma^-} X(\sigma^-)$$

$$\bar\partial X = \bar\partial \tilde X_L = -i \sqrt{\frac{\alpha'}{2}} \sum \tilde\alpha_n \bar z^{-n-1} = \frac{\partial \sigma^+}{\partial z} \partial_{\sigma^+} X(\sigma^+).$$

Then, by miraculous formulas of complex analysis

$$\alpha_n = \sqrt{\frac{2}{\alpha'}} \frac{1}{2\pi} \oint z^n \partial X \, dz$$

$$\tilde\alpha_n = \sqrt{\frac{2}{\alpha'}} \frac{1}{2\pi} \oint \bar z^n \bar\partial X \, d\bar z$$

where integrals are clockwise / anti-clockwise over arbitrary contour encircling origin. (also true on cylinder, as α_n's are conserved charges).

• The whole point of euclidean action, however, is to realize quantization by giving a path-integral representation of correlation functions / vacuum expectation value of (products of) observables.

Def: • An operator is a function on field space that we plan to stick in path-integral.

• A local operator is one that depends only on ~~fields~~ elementary fields (integration variables) and derivatives at a point.

• ~~A primary field / conformal tensor / "good conformal field" is a local operator that transforms as a density of fixed weights under local finite or infinitesimal conformal transformations.~~

Examples: • ~~∂X is a~~ $X(z,\bar{z})$ is a local operator

• $\alpha_n = \frac{1}{2\pi}\sqrt{\frac{2}{\alpha'}} \oint z^n \partial X \, dz$ is an operator

generic symbol: $\mathcal{O}(z,\bar{z})$ for local operator.

- In a conformal field theory, conformal transformation act on field space, leaving action invariant.
- A primary field / conformal tensor / "good conformal field" of weights $(h,\tilde{h})$ is a local ~~observable~~ operator that "looks like a density of these fixed weights", meaning it "transforms under conformal transformations

$$z = F(w)$$

[$h, \tilde{h} \in \mathbb{R}$, not necessarily equal]

as

$$F^{*}\mathcal{O}(z,\bar{z}) = \left(\frac{dw}{dz}\right)^{h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{\tilde{h}} \mathcal{O}(w,\bar{w}).$$

or infinitesimally under

$$z = w + \varepsilon\zeta \qquad \zeta\partial_z \quad (\text{hol.})\ \text{CKV}$$

as

$$L_\zeta \mathcal{O} = \delta_\zeta \mathcal{O} = h\,\partial\zeta\,\mathcal{O} + \zeta\partial\mathcal{O} + \tilde{h}\,\bar{\partial}\bar{\zeta}\,\mathcal{O} + \bar{\zeta}\bar{\partial}\mathcal{O}$$

$$\partial = \frac{\partial}{\partial z} \qquad \bar{\partial} = \frac{\bar{\partial}}{\partial \bar{z}}$$

Example:

- ∂X is primary of weight $(1,0)$

$$\delta_{\ell_n}\partial X = L_{\ell_n}\partial X = \partial\, z^{n+1}\partial X = z^{n+1}\partial^2 X + (n+1)z^n\,\partial X$$

- $T = (\partial X)^2$ wants to be primary of weight $(2,0)$ but fails quantum-mechanically.
- $\partial^2 X$ is not primary.

~~X~~

- Both of these however retain a fixed good behaviour under rescalings and translations, the stress tensor also under special conformal trafo generated by $l_1 = z^2\partial_z$ ("quasi-primary").

Remarks: · A priori, $h, \tilde{h}$ are real and independent from each other. However, $J = l_0 - \bar{l}_0 = -i\partial_\varphi$ generating rotations, σ will transform near $z=0$ as

$$\sigma \to e^{-i(h-\tilde{h})\varphi}\,\sigma$$

and be single-valued only if $h - \tilde{h} \in \mathbb{Z}$.

(for fermionic theories, we'll allow half-integer spin).

("locality / loop & branch cuts are not allowed)

· A primary field which (by equations of motion / upon quantization / under path-integral) satisfies

$$\bar{\partial}\phi = 0$$

is called chiral or holomorphic. It can be expanded in modes

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n \, z^{-n-h}$$

$$\phi_n = \frac{1}{2\pi i}\oint \phi(z)\, z^{n+h-1}\, dz$$

· For a general primary

$$\sigma = \sum_{n,m} \sigma_{n,m}\, z^{-n-h}\, \bar{z}^{-m-\tilde{h}}$$

but residue formula a bit murky.

under conformal transformations, the modes transform as

$$\delta_{l_n} \phi_m = \frac{1}{2\pi i} \oint z^{m+h-1} \left(\cancel{\phi}\, h(n+1) z^n \phi + z^{n+1} \partial \phi \right)$$

$$= \frac{1}{2\pi i} \oint z^{m+n+h-1} \left(h(n+1) - (m+n+h) \right) \phi$$

$$= \left(n(h-1) - m \right) \phi_{n+m}$$

which we have seen as "commutator with L_n" a couple times before.