Lecture 9 BRST cohomology

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Upshot: critical dimension and $a=1$ emerge even more naturally / intrinsically from Virasoro than in OCQ.

Disclaimer: Ideally, one should find / give a first principles, intuitive introduction / motivation physique to this method. For now, we refer to elsewhere, and give a bit of "math" lecture.

① Lie algebra cohomology

- Say $\mathfrak{g}$ is a finite-dimensional Lie algebra (over $\mathbb{R}$, or perhaps $\mathbb{C}$) and (M,ρ) a (complex) representation of $\mathfrak{g}$ ($\mathfrak{g}$-module) and that our problem is to find all $\mathfrak{g}$-invariants

$$M^{\mathfrak{g}} = \{ m \in M,\ \rho(x)\, m = 0 \quad \forall x \in M \}$$

by describing them in a more natural algebraic way as "functorial gadget", such as kernel / image of some map.

- Then, this can be accomplished by noting that the map defining the rep

$$\rho : \mathfrak{g} \to \mathrm{Hom}(M, M)$$

is also a map $\mathfrak{g} \otimes M \to M$ and also a map

$$Q : M \longrightarrow \mathfrak{g}^\vee \otimes M = \mathrm{Hom}(\mathfrak{g}, M)$$

given explicitly by

$$Q(m)(x) = \rho(x)\, m$$

and by which

$$M^{\mathfrak{g}} = \mathrm{Ker}\, Q.$$

<u>Note:</u> - From $\rho : \mathfrak{g} \otimes M \to M$ we can alternatively obtain a map

$$\rho^\vee : M^\vee \to \mathfrak{g}^\vee \otimes M^\vee \qquad \rho^\vee \in M^{\vee\vee} \otimes \mathfrak{g}^\vee \otimes M^\vee$$

which is equal to $Q \in M^\vee \otimes \mathfrak{g}^\vee \otimes M$ if $M^{\vee\vee} = M$, i.e. M finite-dimensional.

- Q can be extended to a differential on a complex (Chevalley-Eilenberg)

$$M \to \mathfrak{g}^\vee \otimes M \to \Lambda^2 \mathfrak{g}^\vee \otimes M \to \Lambda^3 \mathfrak{g}^\vee \otimes M \to \cdots$$

$$Q^2 = 0 \qquad \to \Lambda^i \mathfrak{g}^\vee \otimes M \xrightarrow{Q^i} \Lambda^{i+1} \mathfrak{g}^\vee \otimes M \to \cdots$$

Given explicitly by

$$Q(\lambda)(x,y) = \rho(x)\lambda(y) - \rho(y)\lambda(x) - \lambda([x,y])$$

where $\lambda \in \mathfrak{g}^\vee \otimes M$ and $Q(Q(m)) = 0$ if ρ is a representation.

$$Q(\mu)(x,y,z) = \rho(x)\mu([y,z]) - \rho(y)\mu([x,z]) + \rho(z)\mu([x,y])$$
$$+ \mu(x,[y,z]) \mp \mu(y,[x,z]) + \mu(z,[x,y])$$

$$= (\rho(x)\rho(y) - \rho(y)\rho(x))\lambda(z) \quad - \rho(x)\lambda([y,z])$$
$$- (\rho(x)\rho(z) - \rho(z)\rho(x))\lambda(y)$$
$$+ (\rho(y)\rho(z) - \rho(z)\rho(y))\lambda(x)$$
$$+ \rho(x)\lambda([y,z]) - \rho([y,z])\lambda(x) + \lambda([x,[y,z]])$$

etc. $= 0$ by Jacobi.

will give a physics formula momentarily.

$\Lambda^* \mathfrak{g}^\vee \otimes M = \bigoplus_{i=1}^{\dim \mathfrak{g}^\vee} \Lambda^i \mathfrak{g}^\vee \otimes M$ is called Chevalley-Eilenberg complex. ~~Cohomol~~ with coefficients in M.

$$H^i(\mathfrak{g}, M) = \operatorname{Ker} Q^i / \operatorname{Im} Q^{i-1}$$

Cohomology.

Facts · If G is connected and simply connected Compact Lie gp with Lie algebra $\mathfrak{g}$, then

$$H^*_{dR}(G) = H^*(\mathfrak{g}, \mathbb{R})$$

· $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$, and higher cohomologies measure other things. E.g.

$$H^2(\mathfrak{g}, M) = [\, M \to f \to \mathfrak{g} \,]$$

Lie algebra extensions

· many other interesting. relevant for physics e.g. duality theory. / Hodge theory.

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② Ghosts

Fact. ~~Let~~ If (e_α) is ~~a~~ a basis of $\mathfrak{g}$, with (c^α) the dual basis of $\mathfrak{g}^\vee$ (ghosts, not to be confused with negative norm states) then

$$\cancel{\Lambda^* \mathfrak{g}^\vee = \mathbb{C}[c^\alpha] \text{ with } [c^i, c^j] = 0}$$

· $\Lambda^j \mathfrak{g}^\vee$ has basis $(c^{\alpha_1} \wedge \dots \wedge c^{\alpha_j})$ $\quad \alpha_1 < \dots < \alpha_j$

- $e^\alpha = c^\alpha \wedge \cdot \; : \; \Lambda^i \mathfrak{g}^\vee \to \Lambda^{i+1} \mathfrak{g}^\vee$ satisfy

$$\{c^\alpha, c^\beta\} = 0$$

and $\quad \Lambda^* \mathfrak{g}^\vee = \mathbb{C}[e^\alpha]$

as a supercommutative algebra.

" Defining "anti-ghosts" as contraction or derivation

$$b_\alpha = \iota_{e_\alpha} : \Lambda^i \mathfrak{g}^\vee \to \Lambda^{i-1} \mathfrak{g}^\vee$$

$$b_\alpha (c^{\beta_1} \wedge \cdots \wedge c^{\beta_i}) = \sum_{k=1}^{i} (-1)^{k-1} c^{\beta_1} \wedge \cdots \widehat{\delta c^{\beta_k}} \wedge \cdots \wedge c^{\beta_i} \, \delta_\alpha^{\beta_k}$$

we have $\quad [b_\alpha, b_\beta] = 0 \quad$ and $\quad \{c^\alpha, b_\beta\} = \delta^\alpha_\beta$

- Namely, $\Lambda^* \mathfrak{g}^\vee$ is Fock space for $(\dim \mathfrak{g})$ fermionic creation / annihilation operators c^α, b_β. built on $\Lambda^0 \mathfrak{g}^\vee = \mathbb{C} \cdot v_0$ (vacuum)

<u>Check:</u> If $\quad [e_\alpha, e_\beta] = f_{\alpha\beta}^{\ \ \gamma} e_\gamma \quad$ (structure constants)

~~then~~ and $\quad \rho(e_\alpha) = \rho_\alpha \in \mathrm{Hom}(M, M)$

$Q : \Lambda^* \mathfrak{g}^\vee \to \Lambda^* \mathfrak{g}^\vee \otimes M$ can be written as

$$Q = c^\alpha \rho_\alpha - \frac{1}{2} f_{\beta\gamma}^{\ \ \alpha} c^\beta c^\gamma b_\alpha .$$

$$= c^\alpha \rho_\alpha - \frac{1}{2} f_{\alpha\beta}^{\ \ \gamma} c^\alpha c^\beta b_\gamma$$

$$Q^2 = \underbrace{c^\alpha c^\beta \rho_\alpha \rho_\beta}_{} - \frac{1}{2} \rho_\alpha f_{\beta\gamma}^{\delta} \underbrace{[c^\alpha, c^\beta c^\gamma b_\delta]}_{}$$

$$\left(\begin{array}{l} \frac{1}{2} c^\alpha c^\beta [\rho_\alpha, \rho_\beta] \qquad\qquad\qquad c^\beta c^\gamma \delta^\alpha_\delta \\ = \frac{1}{2} c^\alpha c^\beta f_{\alpha\beta}^{\gamma} \rho_\gamma - \frac{1}{2} \rho_\alpha f_{\beta\gamma}^{\alpha} c^\beta c^\gamma = 0 \end{array} \right)$$

$$+ \frac{1}{4} f_{\alpha\beta}^{\gamma} f_{\delta\varepsilon}^{\zeta} c^\alpha c^\beta b_\gamma c^\delta c^\varepsilon b_\zeta$$

$$= \frac{1}{4} f_{\alpha\beta}^{\gamma} f_{\delta\varepsilon}^{\zeta} c^\alpha c^\beta c^\delta c^\varepsilon b_\gamma b_\zeta \quad \left(\begin{array}{l} = 0 \text{ by symmetry/} \\ \text{antisymmetry } \gamma \leftrightarrow \zeta \end{array} \right)$$

$$\left. \begin{array}{l} + \frac{1}{4} f_{\alpha\beta}^{\gamma} f_{\gamma\varepsilon}^{\zeta} c^\alpha c^\beta c^\varepsilon b_\zeta \\ - \frac{1}{4} f_{\alpha\beta}^{\gamma} f_{\delta\gamma}^{\zeta} c^\alpha c^\beta c^\delta b_\zeta \end{array} \right\} = \quad \left(f_{\delta\gamma}^{\zeta} = - f_{\gamma\delta}^{\zeta} \right)$$

$$\frac{1}{2} f_{\alpha\beta}^{\gamma} f_{\gamma\varepsilon}^{\zeta} c^\alpha c^\beta c^\varepsilon b_\zeta = 0 \quad \text{by Jacobi-identity}$$

for $[[e_\alpha, e_\beta], e_\varepsilon] + [[e_\beta, e_\varepsilon], e_\alpha] + [[e_\varepsilon, e_\alpha], e_\beta] = 0$

$= 0$ on $\Lambda^* \mathfrak{g}^\vee \otimes M$.

For a different point of view, letting

$$\pi_\alpha = - f_{\alpha\beta}{}^{\gamma} c^\beta b_\gamma$$

we have

$$[\pi_\alpha, c^\beta] = - f_{\alpha\gamma}{}^{\beta} c^\gamma \qquad \text{"co-adjoint"}$$

$$[\pi_\alpha, b_\beta] = f_{\alpha\beta}{}^{\gamma} b_\gamma \qquad \text{"adjoint".}$$

$$[\pi_\alpha, \pi_\beta] = \cdots = f_{\alpha\beta}{}^{\gamma} \pi_\gamma .$$

i.e. $\Lambda^* \mathfrak{g}^\vee$ carries representation of $\mathfrak{g}$ (induced from co-adjoint on $\mathfrak{g}^\vee$) and

$$Q = c^\alpha \rho_\alpha + \frac{1}{2} c^\alpha \pi_\alpha \qquad \left(\text{note } \tfrac{1}{2}!\right)$$

. $\frac{1}{2}$ O.K. to ensure $Q^2 = 0$ and, equally crucially

$$[Q, b_\alpha] = \rho_\alpha + \frac{1}{2}\pi_\alpha + \frac{1}{2} c^\beta f_{\beta\alpha}{}^{\gamma} b_\gamma$$

$$= \rho_\alpha + \pi_\alpha .$$

Finally, we record the ghost-number operator (cohomological degree)

$$N^{gh} = c^\alpha b_\alpha$$

$$[N^{gh}, c^\alpha] = c^\alpha \qquad [N^{gh}, b_\alpha] = - b_\alpha$$

and observe that if $\mathfrak{g}$ is (real but presented as) complex Lie algebra with anti-linear involution that real structure is also induced on c_α, b_α.

③ Semi-infinite version

- While the above machinery is overkill for physics in finite-dimensional situations, it is actually useful in applications with infinite-dim. Lie algebras such as Virasoro; which we recall:

$$\mathcal{V} = \underbrace{\mathcal{V}_+}_{\substack{L_n \\ n>0}} \oplus \underbrace{\mathcal{V}_0}_{L_0} \oplus \underbrace{\mathcal{V}_-}_{\substack{L_{-n} \\ n>0}} \oplus \underbrace{c \cdot \mathbb{C}}_{\substack{\text{assume const.} \\ \text{and do not} \\ \text{carry}}}$$

and we are interested in representations in which L_0 is diagonalizable and $L_n \psi = 0$ for n large enough for any state ψ of finite weight.
{eigenvalue: ~~weight~~ (conformal) weight} h

The difference between weight of ψ and highest (lowest) weight is called level. $\Delta = h - h_0$.

- To construct the ghost module in this case, we choose a polarization of $\mathcal{V} \oplus \mathcal{V}^\vee$ adapted to the fact that we do (not) expect to impose $L_n = 0$ strongly (for all n.) only for $n \geq 0$.

"level-wise dual"

Namely, we put

$$\Lambda = \Lambda^*\left(\mathcal{W}_+^\vee \oplus \mathcal{W}_0^\vee \oplus \mathcal{W}_- \right)$$

or in practical terms, fermionic Fock space for

$$\{ c^n, b_m \} = \delta^n_m$$

$$\overset{\shortparallel}{c_{-n}}$$

built on v_0 with $b_n v_0 = 0 \quad \forall\, n \geq 0$

$$c_{-n} v_0 = 0 \quad \forall\, n > 0$$

$$c^{-n}$$

· Then as we saw in HW, Λ indeed carries a representation on $\mathcal{W}$ with

$\underline{n \neq 0:}$ $$L_n^{gh} = \sum_{m \in \mathbb{Z}} (n-m)\, b_{n+m} c_{-m} \quad \left(= -\sum f_{nm}^{k} c^m b_k \right.$$

$\underline{n = 0:}$ $$L_0^{gh} = \frac{1}{2}[L_1, L_{-1}] = \sum_{m \geq 1} m\, b_{-m} c_m + \sum_{m \geq 0} m\, c_{-m} b_m - 1$$

$$L_n^{gh} = :\sum (n-m)\, b_{n+m} c_{-m}: - \delta_{n,0}$$

$: \;\; :$: normal ordering - put all annihilation ops to the right.

of central charge $c = -26$ and $h_0 = -1$ under which

$$[L^{gh}_n, b_m] = (n-m)\, b_{n+m} \qquad (\text{"like the } L_{n+m}\text{"})$$

$$[L^{gh}_n, c_m] = (-2n-m)\, c_{n+m}$$

$$\left([L^{gh}_n, c^m] = -f_{nk}^{\ \ m} c^k \quad \text{checks.} \right)$$

—

In preparation we ~~stiffen~~ grade Λ by ghost number:

$$N^{gh} = \sum_{n>0} \left(c_{-n} b_n - b_{-n} c_n \right) + \tfrac{1}{2}(c_0 b_0 - b_0 c_0)$$

s.t. $[N^{gh}, c_n] = c_n \qquad [N^{gh}, b_n] = -b_n$

$$N^{gh} v_0 = -\tfrac{1}{2}.$$

$\underline{n>0}$ $\quad N^c_n = c_{-n} b_n \qquad N^b_n = b_{-n} c_n$

$$N^{gh} = \sum_{n>0} \left(N^c_n - N^b_n \right) + \tfrac{1}{2}(c_0 b_0 - b_0 c_0)$$

$$L^{gh}_0 = \sum_{n>0} \left(n N^c_n + n N^b_n \right) - 1$$

If (M, L_n, c) is "matter" Virasoro module, we define on

$$\Lambda \otimes M$$

$$Q = c_{-n}L_n + \frac{1}{2} :c_{-n}L^{gh}_n: - \frac{1}{2}c_0$$

$$:c_{-n}L^{gh}_n: = \begin{bmatrix} c_{-n}L^{gh}_n & n \geq 0 \\ L^{gh}_n c_{-n} & n < 0 \end{bmatrix}$$

$$= c_{-n}L_n \cancel{+} - \frac{1}{2} :c_{-n}c_{-m}b_{n+m}:(n-m) - c_0$$

$$:c_{-n}c_{-m}b_{n+m}: = \begin{cases} c_{-n}c_{-m}b_{n+m} & n+m \geq 0 \\ b_{n+m}c_{-n}c_{-m} & n+m < 0 \end{cases}$$

~~and~~ s.t.:

$$[Q, b_m] = L_m + \frac{1}{2}L^{gh}_m + \frac{1}{2}\sum_{n \geq 0}(n-m)c_{-n}b_{n+m} - \frac{1}{2}\sum_{n<0}(n-m)b_{n+m}c_{-n} - \frac{1}{2}\delta_{m0}$$

$$= L_m + L^{gh}_m = L^{tot}_m$$

as desired.

(4) Main claim:

$$Q^2 = 0 \iff c = 26.$$

super Jacobi-identity

<u>Quick argument</u> (for <u>sufficiency</u> only)

① <u>SO</u> $\left[b_m, [L_n^{tot}, Q]\right] = \left[L_n^{tot}, [b_m, Q]\right] + \left[[b_m, L_n^{tot}], Q\right]$

$$= \left[L_n^{tot}, L_m^{tot}\right] - (n-m) L_{n+m}^{tot}$$

$$= \delta_{n+m} \frac{c-26}{12}(n^3 - n) \quad = 0$$

$\Rightarrow$ $[L_n^{tot}, Q]$ does not "depend on c_m" for any m

but since $[N^{gh}, Q] = 1$ $\quad [N^{gh}, L_n^{tot}] = 0$

this implies $[L_n^{tot}, Q] = 0$

②

$$[Q, L_n^{tot}] = [Q, \{Q, b_m\}] = \frac{1}{2}[Q^2, b_m]$$

$$= [Q^2, b_m] - [Q, \{Q, b_m\}]$$

$$= \frac{1}{2}[Q^2, b_m]$$

$\Rightarrow$ Q^2 does not depend on c_m

$\rightsquigarrow$ $Q^2 = 0$

In more detail...

NB: This argument is both necessary & sufficient and the correct one to boot.

$[Q, L_m] = c_{-n}[L_n, L_m]$

$$[Q, L^{gh}_m] = [c_{-n}, L^g_m]L_n + \frac{1}{2}\sum_{n\geq 0}[c_{-n}L^g_n, L^g_m] + \frac{1}{2}\sum_{n<0}[L^g_n c_{-n}, L^g_m] - \frac{1}{2}[c_0, L^g_m]$$

$$= \underbrace{(2m-n)c_{m-n}L_n}_{=(m-n)c_{-n}L_{n+m}} - mc_m + \frac{1}{2}\sum_{n\geq 0} c_{-n}[L^g_n, L^g_m] + \frac{1}{2}\sum_{n>0}[L^g_n, L^g_m]c_{-n}$$

NB: all sums locally finite

$$+\underbrace{\frac{1}{2}\sum_{n\geq 0}(2m-n)c_{m-n}L_n + \frac{1}{2}\sum_{n<0}L^g_n(2m-n)c_{m-n}}$$

$$2\cdot(\cdot) = \sum_{n\geq -m}(m-n)c_{-n}L^g_{n+m} + \sum_{n<-m}L^g_{n+m}(m-n)c_{-n}$$

$$= \sum_{n\geq 0}(m-n)c_{-n}L^g_{n+m} + \sum_{n<0}(m-n)L^g_{n+m}c_{-n}$$

$\pm$ $\underline{m \geq 0}$ $$\sum_{n=-m}^{-1}(m-n)(2n+2m-n)c_m =$$

$$= \sum_{n=1}^{m}(m+n)(2m-n)c_m = \frac{13m^3-m}{6}c_m$$

$+$ $\underline{m<0}$ $$\sum_{n=0}^{-m-1} -(m-n)(2n+2m-n)c_m$$

$$= \sum_{n=0}^{-m-1}(n-m)(2m+n)c_m = \frac{13m^3-m}{6}c_m$$

So $$[Q, L^{gh}_m] = (m-n)c_{-n}L_{n+m} + \frac{1}{2}\sum_{n\geq 0}c_{-n}\left([L^g_n, L^g_m] - L^g_{n+m}(n-m)\right)$$

$$+\frac{1}{2}\sum_{n<0}\left([L^g_n, L^g_m] - (n-m)L^g_{n+m}\right)c_{-n}$$

$$+\frac{13}{12}(m^3-m)c_m$$

$$= (m-n)c_{-n}L_{n+m} + \frac{26}{12}(m^3-m)c_m$$

$$\rightsquigarrow \left[Q, L_m + L_m^{gh} \right] = \frac{26-c}{12} (m^3 - m) c_m .$$

$$[Q, c_m] = \frac{1}{2} \sum_{n \geq 0} c_{-n} (-2n-m) c_{n+m} - \frac{1}{2} \sum_{n<0} (-2n-m) c_{n+m} c_{-n}$$

$$= \frac{1}{2} (2n+m) c_{n+m} c_{-n}$$

$$[Q,Q] = \left[Q, c_{-m} L_m + \frac{1}{2} :c_{-m} L_m^{g}: - \frac{1}{2} \delta \right] = \overbrace{\frac{1}{2}(2n-m) c_{n-m} c_{-n} L_m}^{\frac{1}{2}(n-m) c_{-m} c_{-n} L_{n+m}} - c_{-m} c_{-n} [L_n, L_m]$$

$$+ \frac{1}{2} \sum_{m \geq 0} \left(\frac{2n-m}{2} c_{n-m} c_{-n} L_m^{g} - (m-n) c_{-m} c_{-n} L_{n+m} - \frac{26}{12} (m^3 - m) c_{-m} c_m \right)$$

$$+ \frac{1}{2} \sum_{m<0} \left(L_m^{g} \frac{2n-m}{2} c_{n-m} c_{-n} + (m-n) c_{-n} L_{n+m} c_{-m} + \frac{26}{12} (m^3 - m) c_m c_{-m} \right)$$

$$- \frac{1}{2} n c_n c_{-n}$$

$$= c_{-n} c_{-m} \left([L_n, L_m] - (n-m) L_{n+m} - \frac{26}{24} \delta_{n+m} (m^3 - m) \right)$$

$$+ \frac{1}{4} \sum_{m \geq 0} (2n-m) c_{n-m} c_{-n} L_m^{g}$$

$$+ \frac{1}{4} \sum_{m<0} (2n-m) L_m^{g} c_{n-m} c_{-n}$$

Note at this point all sums are a priori locally finite. For the first we only have a contribution when $n = -m$ either going to infinity $c_{-n} c_n$ kills any state for n large enough.

For the last two: for each state there is a maximum (minimum) m from which on L_m^{g} will kill the state. And for each m there are max/min values of n that will kill the state. So overall ~~there are finite regions in n-m space~~ the sums are restricted to finite regions in n-m space.

So we can shift m index and recalling

$$L_n^{gh} = \sum_{n+k\geq 0} -(n-k)\, c_{-k} b_{n+k} + \sum_{n+k<0} (n-k)\, b_{n+k} c_{-k} - \delta_{n0}$$

we find: $4\times(\,)$:

$$\sum_{\substack{n+m+k\geq 0\\ m+n\geq 0}} f_{nmk}\, c_{-n} c_{-m} c_{-k} b_{n+m+k} + \sum_{\substack{n+m+k\geq 0\\ m+n<0}} f_{nmk}\, c_{-k} b_{n+m+k} c_{-n} c_{-m}$$

$$+ \sum_{\substack{n+m+k<0\\ n+m\geq 0}} - f_{nmk}\, c_{-n} c_{-m} b_{n+m+k} c_{-k} + \sum_{\substack{n+m+k<0\\ n+m<0}} - f_{nmk}\, b_{n+m+k} c_{-k} c_{-n} c_{-m}$$

$$- \; 2n\, c_n c_{-n}$$

where all sums are locally finite. and $f_{nmk} = (n-m)(n+m-k)$ satisfies $f_{nmk} + f_{mkn} + f_{knm} = 0$ by Jacobi-identity so that what remains is

$$\sum_{\substack{n+m+k\geq 0\\ m+n<0}} f_{nmk} \left(\delta_{m+k}\, c_{-k} c_{-m} - \delta_{n+k}\, c_{-k} c_{-n} \right)$$

$$+ \sum_{\substack{n+m+k<0\\ n+m\geq 0}} \left(- f_{nmk}\, \delta_{n+k}\, c_{-n} c_{-k} + f_{nmk}\, \delta_{m+k}\, c_{-m} c_{-k} \right)$$

$$- \; 2n\, c_n c_{-n}$$

$$\sum_{\substack{n\geq 0\\ m+n<0}} (n-m)(2m+n)\, c_m c_{-m} - \sum_{\substack{m\geq 0\\ n+m<0}} (n-m)(2n+m)\, c_n c_{-n}$$

$$= 2 \sum_{m<0} c_m c_{-m} \underbrace{\sum_{n=0}^{-m-1} (n-m)(2m+n)}_{\frac{13m^3-m}{6}}$$

$$+ \sum_{\substack{n<0\\ n+m\geq 0}} -(n-m)(2n+m)\, c_{-n} c_n + \sum_{\substack{m<0\\ n+m\geq 0}} (n-m)(2m+n)\, c_{-m} c_m$$

$$= 2 \sum_{m\geq 0} c_{+m} c_{-m} \underbrace{\sum_{n=-m}^{-1} (n-m)(2m+n)}_{\frac{13m^3-m}{6}}$$

So $\overline{\frac{1}{4}}\cdot = \sum_m c_m c_{-m} \frac{13}{12}(m^3-m)$

For a grand total of

$$[Q,Q] = \sum_n c_{-n} c_n \frac{c-26}{12}(n^3-n)$$

$$= 0 \iff c = 26.$$

Lecture 10 No-ghost theorem

① Claims

To utilize the semi-infinite Lie algebra cohomology for comparison between $\mathcal{H}_{OCQ}$ and $\mathcal{H}_{LCG}$, we consider on

$$\mathcal{G} = \Lambda \otimes \mathcal{F}^{l.c.} \otimes \mathcal{H}^{\perp}$$

$$\underbrace{\mathcal{F}^{+}_{p_+} \otimes \mathcal{F}^{-}_{p_-}}_{\mathcal{F}}$$

transverse matter, unitary rep of Vir with $c=24$ and (say) discrete spectrum of $L_0^{\perp}$.

$$Q = c_{-n}\big(\underbrace{L_n^{l.c.} + L_n^{\perp}}_{L_n}\big) + \frac{1}{2} : c_{-n} L_n^{gh} : - \frac{1}{2} c_0 \qquad Q^2 = 0.$$

w.r.t. ghost number

$$N^{gh} = \sum_{n>0} \underbrace{c_{-n} b_n}_{N_n^c} - \underbrace{b_{-n} c_n}_{N_n^b} + \frac{1}{2}(c_0 b_0 - b_0 c_0)$$

we have a decomposition

$$\mathcal{G} = \bigoplus_{i \in \mathbb{Z}+\frac{1}{2}} G^i \qquad v_0 \otimes \mathcal{F}^{l.c.} \otimes \mathcal{H}^{\perp} \subset G^{-\frac{1}{2}}.$$

where all the action is. (but note $G^{-\frac{1}{2}}$ contains many more states)

and $[N^{gh}, Q] = Q$.

Rough Claim: Up to zero modes (c_0, b_0, L_0, p_+, p_-), Λ cancels $\mathcal{F}^{l.c.}$ in cohomology (so $\mathcal{H}_{LCG} \sim \mathcal{H}^{\perp}$), while alternatively, $\operatorname{Ker} Q = \mathcal{F}_{phys}$, $\operatorname{Im} Q = \mathcal{F}_{null}$, so $H^*(\mathcal{G}) = \mathcal{H}_{OCQ}$.

<u>More precisely</u>, we claim:

(i) $H^i(\mathfrak{g}) = 0$ unless $i = -\frac{1}{2}$ or $+\frac{1}{2}$

$p^+ \neq 0$ $H^{-\frac{1}{2}}(\mathfrak{g}) \cong \mathcal{H}_{LCG} = \{ \psi \in \mathcal{H}^\perp, (L_0^\perp - 1)\psi = 2\alpha' p^+ p^- \psi \}$

No! (possibly empty most of the time, if $L_0^\perp$ has discrete spectrum)

$c_0 : H^{-\frac{1}{2}} \to G^{\frac{1}{2}}$ induces isomorphism $H^{-\frac{1}{2}} \to H^{\frac{1}{2}}$

(ii) The map $\mathcal{F} \to G^{-\frac{1}{2}}$ $\psi \mapsto v_0 \otimes \psi \in G^{-\frac{1}{2}}$

induces isomorphism $\mathcal{H}_{OCQ} = H^{-\frac{1}{2}}$

i.e.

$$(L_n - a\delta_n)\psi = 0 \Rightarrow Q(v_0 \otimes \psi) = 0$$

and $\|\psi\|^2 = 0 \iff \psi = Q\chi$.

both directions!

② Inner products

~~For this to fully imply "no ghosts", it is important~~

→ We have two representations of the same cohomology, but for this to fully imply "no ghosts", it is important that the inner product be the same, whether induced from $\mathcal{F}$ via $\mathcal{H}_{OCQ}$ or only from $\mathcal{F}^\perp$ (via $\mathcal{H}_{LCG}$). This is ensured by defining an inner product on $\mathfrak{g}$ wrt which $Q = Q^\dagger$, such that well-defined on H^* and which agrees with the two others in the presentations.

· The natural way to ensure $Q^{\dagger}=Q$, as alluded to before, is to give c_n's (and b_n's) the same reality as L_n, i.e.

$$c_n^{\dagger} = c_{-n} \qquad b_n^{\dagger} = b_{-n}$$

· This implies, N.B., that creation & annihilation ops. are not adjoint of each other. More precisely

$$N^{gh\,\dagger} = - N^{gh}$$

implies that

$$\Lambda^i \perp \Lambda^j \quad \text{unless } i=-j$$

which exhibits convenience of half-integer grading.

· We normalize inner product by

$$\langle \underbrace{V_0}_{-\frac{1}{2}} | \underbrace{c_0 V_0}_{+\frac{1}{2}} \rangle = 1 \qquad \text{which implies the rest.}$$

~~Then~~ if On $\mathcal{G}$, we product with inner product on $\mathcal{F}$.

· Then, if $Q\psi = 0$, $\varphi = Q\chi$

$$\langle \varphi, \psi \rangle = \langle Q\chi, \psi \rangle = \langle \chi, Q^{\dagger}\psi \rangle$$

$\Rightarrow$ $\langle \cdot | \cdot \rangle$ well-defined on cohomology.

Preliminary remark about $\mathcal{H}_{OCQ} = \mathcal{H}_{BRST}$

Let's ~~If~~ ~~we~~ temporarily assume (i) and moreover that the inner product on ~~…~~ $H^{-1/2}$ by

$$(\varphi, \psi) = \langle \varphi | c_0 | \psi \rangle$$

agrees with the one on $\mathcal{H}_{LCG}$ (and is therefore pos. def.)

Then we note that if $\psi \in \mathcal{F}_{phys}$ $(L_n - \delta_{n,0})\psi = 0$ $n \geq 0$ then

$$Q(v_0 \otimes \psi) = \sum_{n \geq 0} c_{-n} v_0 \otimes (L_n - \delta_{n,0}) \psi = 0$$

Moreover

$$\|\psi\|^2 = \langle v_0 \otimes \psi | c_0 | v_0 \otimes \psi \rangle = (v_0 \otimes \psi, v_0 \otimes \psi) = 0$$

$$\Leftarrow \quad v_0 \otimes \psi = Q(-)$$

This shows:

$\mathcal{H}_{OCQ} \longrightarrow \mathcal{H}_{BRST}$ is well-defined and injective.

(surjectiveness later) or not

③ Aside – but important!

$$\mathcal{F}_{l.c.} = \tilde{\mathcal{F}}^{l.c.}$$

- This oddity of Λ is very similar to that of $\tilde{\mathcal{F}}^{l.c.}$; in that annihilation operators for α^-_{-n} are α^+_n and vice-versa.

- More succinctly, defining

$$N^{l.c.} = \sum_{n>0} \frac{1}{n}\Big(\underbrace{\alpha^+_{-n}\alpha^-_n}_{-N^+_n} - \underbrace{\alpha^-_{-n}\alpha^+_n}_{+N^-_n}\Big)$$

~~we have~~ $$[N^{l.c.}, \alpha^\pm_{\pm n}] = \mp \alpha^\pm_n$$

$$\left([\alpha^+_n, \alpha^-_m] = -n\,\delta_{n+m}, \qquad \eta^{+-} = \eta^{-+} = -1 \right)$$

we can decompose $\tilde{\mathcal{F}}^{l.c.} = \bigoplus_i \mathcal{F}^{l.c.\,i}$

and because of $N^{l.c.\,+} = -N^{l.c.}$ we have

$$\mathcal{F}^{l.c.\,i} \perp \mathcal{F}^{l.c.\,j} \quad \text{unless } i = -j.$$

→ so cancelling $\mathcal{F}^{l.c.} \leftrightarrow \Lambda$ seems plausible.

(Note however zero modes are different.)

→ our complex is in fact bi-graded

$$\mathcal{G} = \bigoplus_{\substack{i \in \mathbb{Z}+\frac{1}{2} \\ j \in \mathbb{Z}}} G^{i,j}$$, which cries out for spectral sequence.

(4) Zero modes

· There is in fact a third compatible grading by the Virasoro weight (eigenvalue of $L_0 = L_0^{S^1} + L_0^{gh} + L_0^{x}$; diagonalizable). $\mathfrak{g} = \bigoplus \mathfrak{g}_h = \bigoplus G_h^{ij}$

· Since $[Q, L_0] = 0$ we can compute H^* for each h separately, and since $\{Q, b_0\} = L_0$ we in fact have

$$H^*(\mathfrak{g}) = H^*(\mathfrak{g}_0).$$

—

· Because b_0, c_0 form fermionic ann/creation pair, we can decompose $\mathfrak{g}$ and each $\mathfrak{g}_h$ (b/c $[L_0, b_0] = 0$) in $\mathrm{Ker}\, c_0 + \mathrm{Ker}\, b_0$.

$$\mathfrak{g}_h = \mathrm{Ker}\,c_0 \;\underset{c_0}{\overset{b_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\oplus}}}}\; \mathrm{Ker}\,b_0$$

where b_0, c_0 are isomorphisms.

~~On $\mathfrak{g}_0$, we have $\{Q, b_0\} = L_0 = 0$, and therefore~~

$$\cancel{b_0 : H^*(\mathfrak{g}_{h}) \longrightarrow H^*(\mathfrak{g}_{h})}$$

$\mathrm{Ker}\, b_0$

- $b_0 : \mathcal{Y}_{h\uparrow} \overset{c_0}{\underset{\longrightarrow}{\longleftarrow}} \mathcal{Y}_{h\downarrow}$ is isomorphism of vectorspaces $\forall h$.

- $[Q, b_0] = L_0 = 0$ on $\mathcal{Y}_0$, so b_0 is in fact iso of complexes at $h=0$

$\Rightarrow \quad b_0 : H^*(\mathcal{Y}_{0\uparrow}) \longrightarrow H^*(\mathcal{Y}_{0\downarrow})$

is an isomorphism.

$\rightarrow$ it is sufficient to calculate

$$H^*(\mathcal{Y}_{0\downarrow}) = \bigoplus_{i,j} {}^0 G^{ij}_{0\downarrow})$$

⑤ Main calculation

While $[N^{gh}, Q] = 0$, this is not true of $N^{l.c.}$

however, it is easy to see that Q changes $N^{l.c.}$ by at most ± 1. We write

$$Q = Q_1 + Q_0 + Q_{-1} \quad \text{where } [N^{l.c.}, Q_j] = j\, Q_j$$

where

$$Q_1 = -\sum_{n \neq 0} c_{-n} \; \alpha_0^+ \alpha_n^-$$

(all other terms contain either + and − oscillators $\alpha_p^+ \alpha_{n-p}^-$ or none at all

$\alpha_0^+ \alpha_0^-$)

$\alpha_0^+ = \sqrt{2\alpha'}\, p^+$

So we calculate cohomology of Q_1 first.

$$Q_1 = -\sqrt{2\alpha'}\, p^+ \sum_{n\neq 0} c_{-n}\alpha^-_n$$

easy to see (e.g. degree reasons or direct calculation) $Q_1^2 = 0$.

(Similar expression for Q_{-1}, but Q_0 is all the messy rest).

Assuming $p^+ \neq 0$, we define

$$R = \frac{1}{\sqrt{2\alpha'}\, p^+} \sum_{m\neq 0} \alpha^+_{-m} b_m$$ (a kind of c/a dual of Q_1). creation/annihilation

Then

$$[Q_1, R] = \sum_n n\, c_{-n} b_n - \alpha^+_{-n}\alpha^-_n$$

(normal ordering required for $n<0$!)

$$= \sum_{n>0} n\, c_{-n} b_n + n\, b_{-n} c_n - \alpha^+_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^+_n$$

$$= \sum_{n\geq 1} n\left(N^c_n + N^b_n + N^+_n + N^-_n\right)$$

$$=: S'_0 \quad \text{- a part of } L_0$$

- counts the level in ghosts & light ~~like~~ cone oscillators

Then S'_0 is diagonalizable and Q_1 invariant (since Q_1 exact

general principle $\Rightarrow$ $H^*(G^{ij}_{0t}, Q_1) = H^*(G^{ij}_{0t} \cap \operatorname{Ker} S_0, Q_1)$

Idea of spectral sequence: calculate cohomology of $Q = Q_1 + Q_0 + Q_{-1}$ by first calculating cohomology of Q_1, then correct for the rest.

- It is easy to see that $Q_1 \big|_{\mathrm{Ker}\, S_0} \equiv 0$ identically (one of c_{-n} α_n^- is annihilation operator.)

$\Rightarrow$ $H^*(\mathcal{B}^{i,j}_{0S}, Q_1) = \mathrm{Ker}\, S'_0$ in degree $i = -\frac{1}{2}$ $j=0$

$= \mathcal{V}_0 \otimes \mathcal{H}_{LCG}$ ~~correctly~~

- Also note that $Q_1^\dagger = Q_1$ and that

$(\psi, \varphi) = \langle \psi, c_0 \varphi \rangle = \langle \varphi, \varphi \rangle_{LCG}$ under this isomorphism.

⑥ The good stuff

- The fact that $Q_1|_{\mathrm{Ker}\, S_0} = 0$ follows more abstractly from $[Q_1, S_0] = 0$, that Q_1 has ghost number 1, while $\mathrm{Ker}\, S_0$ is completely in ghost number $-\frac{1}{2}$.

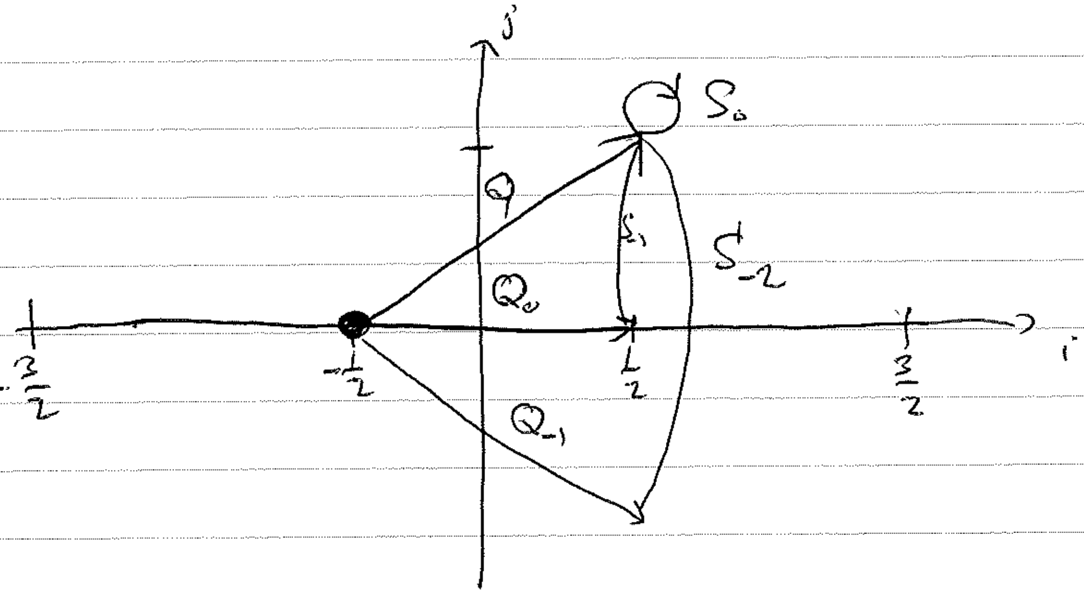
$N^g = \frac{1}{2}$

$S_0 \psi = 0 \Rightarrow Q_1 S_0 \psi = S'_0 Q_1 \psi = 0$

$N^g = -\frac{1}{2}$ $\Rightarrow Q_1 \psi = 0$ invertible in this ghost degree

Now return to $Q = Q_1 + Q_0 + Q_{-1}$ and consider

$$S := [Q, R] = S_0 + S_{-1} + S_{-2}$$



⇒ For same reason as before

$$H^*(G^{i,j}_{\alpha}, Q) = H^*(G^{i,j}_{\alpha\beta\gamma} \cap \mathrm{Ker}\, S, Q)$$

and $Q|_{\mathrm{Ker}\, S} = 0$ follow if we can establish the claim

$$\mathrm{Ker}\, S \cong \mathrm{Ker}\, S_0 \quad \text{and } \mathrm{Ker}\, S \text{ is all in ghost degree } -\tfrac{1}{2}$$

Pf: $S_0 \psi_0 = 0$, let $U = S - S_0 = S_{-1} + S_{-2}$

def $\psi = \underbrace{\psi_0}_{N^{gc} = 0} - \underbrace{S_0^{-1} U \psi_0}_{N^{gc} = -1} + S_0^{-1} U S_0^{-1} U \psi_0 - \dots$

S_0 invertible here

. Then up to a mild analytic issue,

$$S'\psi = 0.$$

. Conversely, let $S'\psi = 0$,

$$\psi = \psi_n + \psi_{n-1} + \dots$$

Then $S_0 \psi_n = 0 \Rightarrow n = 0$ and if $\psi_0 = 0$ all $\psi_{-1}, \psi_{-2},$ etc are also 0.

In other words, $\psi \in \mathrm{Ker}\, S'$ is fully determined by its $N^{l.c.} = 0$ component. The map

$$\mathrm{Ker}\, S' \ni \psi \longrightarrow \psi_0 \in \mathrm{Ker}\, S_0$$

is one-to-one, onto.

Moreover, because of $N^{l.c.\dagger} = -N^{l.c.}$ as observed before, the inner product is given by that on $N^{l.c.} = 0$ component.

This proves all claims in (i).

For (ii) remains to show surjectivity of $\mathcal{H}_{OCQ} \to \mathcal{H}_{BRST}$. In fact (see Polchinski) one can show that $\underbrace{\text{state } \psi}_{\text{representative of cohomology class.}}$ attached to ψ_0 in above fashion is in the image of $\mathcal{H}_{OCQ}$.

- What Polchinski does not discuss, but can be found in other sources, is what happens when both $p^+ = p^- = 0$

→ It appears that in that case,

$$H^i(\mathcal{O}_J) = 0 \quad \forall\, i \text{ except } i = \pm 3/2, \text{ when it is one-dimensional}$$

$$\begin{cases} H^{-\frac{3}{2}}(\mathcal{O}_J) = \mathbb{C} \cdot b_{-1} v_0 \\ H^{\frac{3}{2}}(\mathcal{O}_J) = \mathbb{C} \cdot c_{-1} v_0 \end{cases}$$

"the string vacuum".

(In other formalisms, one presumably has to add this by hand.)