Lecture 7 Constraints & String spectrum

Recall goal as advertised: Describe / understand / define string theory as a relativistic quantum mechanical theory of massless fundamental strings.
Technically, this means we desire Hilbert space

$$\mathcal{H} = \mathcal{H}_{string}$$

with

1) unitary representation of Poincaré symmetry, the irreducible pieces of which we want to interpret as elementary particles
2) a unitary S-matrix

$$\underbrace{\bigoplus_{N=0}^{\infty} \mathcal{H}_{string}^{\otimes N}}_{=\mathcal{H}_{many\ strings}} \longrightarrow \mathcal{H}_{many\ strings}$$

describing their interactions.

Note: • These are the same wishes as in QFT, where one learns that, supplemented with locality, they are quite restrictive and imply e.g. existence of anti-particles, gauge principle, anomalies. (in fact, S-matrix ???)

• Strings fundamentally jeopardize locality; instead there are other principles that at the end of day make them even more restrictive. In the meantime, ...

- at low energies (and there is always time; mass scale $\sim M_{Pl} \sim 10^{19}$ GeV), only finitely many species are relevant (mass spectrum is discrete), locality is restored, and particle interactions have to be describable by "effective" QFT of some sort.
- This kind of "stringy correspondence principle" says that rather than being superseded, QFT is a pre-theory for strings, necessary for its interpretation. Also, we can appeal to our knowledge of QFT for certain consistency considerations.

- We derive $\mathcal{H}_{string}$ to be constructed from ingredients collected last week, including
  - D sets of canonical variables x^μ, α^μ_n, $(\tilde{\alpha}^\mu_n)$.
  - represented on their natural Fock space

$$\mathcal{F} = \bigotimes_{\mu=0}^{D} \mathcal{F}^\mu$$

  - because $\mathcal{F}$ has natural action of $\mathcal{P}$ with

$$J^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu + i \sum \frac{\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n}{n} \quad (+ \sim)$$

  - $\mathcal{F}$ is (at least (at best)) a pseudo pre-Hilbert space.

indefinite $\|\cdot\|$ incomplete

by "imposing the Virasoro constraints $L_n = 0$ on physical process".

$$L_n = T \int e^{in\sigma} \partial_- X \partial_- X = \frac{1}{2} \sum \alpha^{\mu}_{-p} \alpha_{n+p\,\mu}$$

which on $\mathcal{F}$ (are well-defined and) satisfy

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}$$

with $c = D$. (N.B. independent of signature).

- physically, central charge c is "anomaly": ~~failure~~ mild failure of classical symmetry
- mathematically, c is ~~[illegible]~~ linearly independent basis element of an ∞-dim Lie algebra.

Today: First go; next time: second attempt; next week: the real deal.

0) The rules of the game

Name: "Old covariant approach", OCQ

Two seasoned players: worldsheet / spacetime
Virasoro / Poincaré

$$[L_n, P^\mu] = [L_n, J^{\mu\nu}] = 0 \qquad \text{(not completely obvious)}$$

$\rightsquigarrow$ in principle, whatever we do, we should get something reasonable.

Principle: unitarity. Prize: D

① Firstplay (worldsheet)

· The representation theory of the Virasoro algebra is very interesting.(*) Here are some totally elementary considerations, based on the unitarity condition

$$L_n^\dagger = L_{-n} \quad \left(\begin{array}{c}\text{reality of worldsheet, worldsheet reparametrization} \\ \text{energy-momentum tensor}\end{array}\right)$$

Say $\mathcal{H}$ is some unitarity representation of Vir with fixed (real) value of central charge, $\psi \in \mathcal{H}$. Then, by considering

$$\| L_{-n}\psi \|^2 = \langle L_{-n}^\dagger \psi | L_{-n}\psi \rangle = \langle \psi | L_n L_{-n} | \psi \rangle =$$

$$= \| L_n \psi \|^2 + 2n \langle \psi | L_0 | \psi \rangle + \frac{c}{12}(n^3 - n) \langle \psi | \psi \rangle$$

we learn: If $L_n|\psi\rangle = 0$ $\forall n$ at $c \neq 0$, by considering $n \neq 0, \pm 1$, that $\psi = 0$. (the ~~...~~ trivial module is not unitary at $c \neq 0$).

→ For string theory, ($c = D$) we (have to) say it's enough if $\langle \psi | L_n | \varphi \rangle = 0$ for any pair of non-zero states. For this it suffices that

$$L_n |\psi\rangle = 0 \qquad \forall n \geq 0.$$

(Since $L_3 \sim [L_2, L_1]$ etc., it's enough to require $L_0, L_1, L_2 = 0$)
→ perhaps don't say this here!)

(*) and, as I mentioned, the general theory is very relevant for "other backgrounds" of string theory.

- Somewhat more generally, one is primarily interested in representations in which
  - L_0 is diagonalizable $L_0 = L_0^+$ ~~[illegible]~~
    (eigenvalues are called conformal weight)
  - $L_n |\varphi\rangle = 0$ for all eigenvectors of L_0, n large enough.
    ($\rightsquigarrow$ highest weight reps, Verma modules etc.)
- For such reps, $c < 0$ would imply
  $\| L_{-n} \varphi \| < 0$ for n large enough, non unitary
  Luckily in string theory $c = D > 0$.
- $\| L_{-1} \varphi \|^2 = 2 \langle \varphi | L_0 | \varphi \rangle$
  $\rightsquigarrow$ conformal weights are positive, though if
  $L_0 |\varphi\rangle = 0 \Rightarrow L_{-1} \varphi = 0$, which appears one too many for invariance.

$\longrightarrow$ We relax the physical state condition to

$$L_n |\varphi\rangle = 0 \qquad n > 0$$

$$(L_0 - a) |\varphi\rangle = 0$$

where $a \geq 0$ is some fixed constant.

- This relaxation also does justice to the fact that during quantization, L_0 was subject to ordering ambiguity, so a is also known as "normal ordering constant."

- In the presence of left- and right-movers, we require $\tilde{a} = a$. This is because

$$L_0 - \tilde{L}_0 = \partial_{\sigma^-} - \partial_{\sigma^+} \sim \partial_\sigma$$

generates rigid rotations of closed string, which is not supposed to be anomalous.

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② Second play (spacetime)

- As remarked before, the import of the constraint imposed by

$$L_0 = \underbrace{\frac{1}{2}\alpha_0^2}_{-A^2/2} + \underbrace{\sum_{p=1}^{\infty} \alpha_{-p} \alpha_p}_{N}$$

is to select, inside

$$\mathcal{F} = L^2(\mathbb{R}^{d+1}) \otimes \bigotimes_{\mu=0}^{d} \mathcal{F}^{\mu}_{\alpha}.$$

states with a specific relation between mass $M^2 = \frac{A^2}{2\alpha'}$ open, $M^2 = \frac{2}{\alpha'} A^2$ (closed string) and total oscillator energy or occupation number, or level

$$N = \sum_{p=1}^{\infty} \alpha_{-p}\alpha_p = \sum_{p=1}^{\infty}\sum_{\mu=0}^{d} p\, a^{\mu +}_p a_{p\mu} \geq 0$$

Spectrum of $N = \mathbb{N}_0$, but highly degenerate.

$$A^2 = N - a = \tilde{N} - \tilde{a}$$

$$\Rightarrow \quad N = \tilde{N} \quad , \text{ so-called level matching.}$$

- Recalling HWs, we see that what remains to be done ⊗ is to decompose each level ⊗ into irreducible representations of the corresponding little group, to identify particle content of the string. ⊗ Then, use knowledge of QFT to address each such irrep by some (local) field multiplet in spacetime. ~~(local)~~

- Level 0: unique ground state $|0, p\rangle$ scalar particle for each p. of mass

$$A^2 = -2a \leq 0$$

⊗ subject to $L_1 = L_2 = 0$.

- Clearly $L_n = 0$ for $n>0$ on this state (though not for $n<0$!) and $\langle p | p' \rangle = \delta(p-p')$ is positive definite.

- Since ~~for~~ $a>0$ $M^2<0$ should give us pause — a tachyon or tachyonic field in spacetime. However, from unitarity pt. of view this is fine. Little gp is $SO(1, d-1)$, standard momentum:

$$(0, M, \dots 0)$$

and trivial rep is precisely the only unitary rep finite-dim. of this gp.

- Then however, states at <u>level 1</u>, which are of the form

$$\alpha^{\mu}_{-1} |p\rangle \qquad \mu = 0 \dots D \qquad \text{(open string)}$$

$$\alpha^{\mu}_{-1} \alpha^{\nu}_{-1} |p\rangle \qquad \mu, \nu = 0 \dots D \qquad \text{(closed string)}$$

have $A^2 = 2(1-a)$, to satisfy $L_0 = a$, transform in non-trivial f.d. representation of little group ~~(physical L_1 L_2 are give at not two linearly independent conditions, for $D>3$)~~ for $D>3$, as L_1, L_2 give at most two ~~indep~~ l.i. conditions. So already from spacetime unitarity pt of view we see that $\underline{\underline{a \leq 1}}$.

More precisely, consider (open string only) the general state at level one:

$$|\varepsilon,p\rangle = \varepsilon_\mu \alpha^\mu_{-1} |p\rangle \qquad \varepsilon_\mu \in \mathbb{R}^{1,d}$$

$$L_1 |\varepsilon,p\rangle = \alpha_1 \cdot \alpha_0 \; \varepsilon_\mu \alpha^\mu_{-1} |p\rangle$$

$$\propto \varepsilon_\mu p^\mu |p\rangle \stackrel{!}{=} 0 \qquad \varepsilon \in p^\perp$$

$$(L_2 |\varepsilon,p\rangle = 0 \quad \text{trivially.})$$

while $\| |\varepsilon,p\rangle \|^2 = \varepsilon^2 = \varepsilon^\mu \varepsilon_\mu$.

Then, if $a > 1$: $\varepsilon \in p^\perp = \mathbb{R}^{1,d-1} \subset \mathbb{R}^{1,d}$

→ there are $d-1$ ~~sta~~ physical states with positive norm, and one with negative norm (a "ghost").

If $0 \leq a < 1$: ~~$\varepsilon \in p^\perp$~~ $p^2 < 0$

$\varepsilon \in p^\perp = \mathbb{R}^d$; and there are d states with positive norm, in fundamental (vector) rep of little gp $SO(d)$.

– a massive vector particle in spacetime, described in QFT e.g. by Proca action

$p = (1, 1, 0, \dots 0)$

- Most interestingly when $\underline{\underline{a=1}}$, $p^2=0$ $\varepsilon \in p^\perp = \mathbb{R}^d$ with degenerate bilinear form, there are
  - $(d-1)$ states with positive norm in fundamental representation of little gp $SO(d-1)$, amounting to physical states of massless vector field (Maxwell action with $U(1)$ gauge symmetry)
  - 1 zero norm state $\varepsilon \propto p$ which we "discard" not as per se unphysical, but rather as so-called "null state".

Note: This construction is similar to that arising in the Gupta-Bleuler quantization of Maxwell theory, in which (typically) Coulomb gauge $\partial_\mu A^\mu$ is imposed "weakly", and states satisfy $p_\mu \varepsilon^\mu = 0$, ~~null~~ $\varepsilon^\mu \wedge p^\nu$ are null.

- From $L_1 |p\rangle = \alpha_0 \cdot \alpha_{-1} |p\rangle \sim p \cdot \alpha_{-1} |p\rangle$ we learn that null states are in fact orthogonal to all physical states which (after a while, $d=26$) ~~which after a~~ implies that they decouple from physical processes (as long as gauge invariance is maintained).
- This illustrates a tight connection between worldsheet and spacetime gauge invariance, which is discussed at some length in the textbooks.

- For the closed string physical states at (left/right) level 1 are subspace

$$\varepsilon_{\mu\nu}\, \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |p\rangle \qquad p^2 = \frac{4}{\alpha'}(a-1)$$

with $p^{\mu}\varepsilon_{\mu\nu} = \varepsilon_{\mu\nu}p^{\nu} = 0$ ("linear map $p^{\perp} \to p^{\perp}$")

- For $a > 1$ there are ghosts, while for $a = 1$, null states of the form

$$\varepsilon_{\mu\nu} = p_{\mu} a_{\nu} \ / \ a_{\mu} p_{\nu} \qquad (a \in p^{\perp})$$

$2d-1$

$$\|\cdot\| \sim \varepsilon_{\mu\nu}\varepsilon^{\mu\nu}.$$

$d^2 - 2d + 1 = (d-1)^2$

ignoring those, the remaining states decompose in irreps of $SO(d-1)$ as

"trace" ($\varepsilon \sim id$)	1 state
symmetric traceless	$\frac{(d-1)d}{2} - 1$
antisymmetric	$\frac{(d-1)(d-2)}{2}$

$\rightarrow$ "string theory contains gravity"

③ Half-time We define

$$\mathcal{F}_{phys} = \{ \psi \in \mathcal{F} \mid L_1\psi = 0 \;\; L_2\psi = 0 \;\; (L_0 - a)\psi = 0 \}$$
$$(\Rightarrow L_n\psi = 0 \;\; \forall n)$$

$$\mathcal{F}_{spurious} = \mathcal{F}_{phys}^{\prime \perp} =$$

$$= \{ \chi \in \mathcal{F},\; (L_0 - a)\chi = 0 \quad \langle \chi | \psi \rangle = 0 \quad \forall \psi \in \mathcal{F}_{phys} \}$$

$$\mathcal{F}_{null} = \mathcal{F}_{phys} \cap \mathcal{F}_{spurious}$$

$$\mathcal{H}_{OCQ} = \mathcal{F}_{phys} / \mathcal{F}_{null}$$

the idea being that $\langle \cdot | \cdot \rangle$ restricts / induces a positive definite inner product on $\mathcal{H}_{OCQ}$, wrt which it can then be completed to a Hilbert space.

(4) Second half

(push the program to discover the critical dimension)

The general open string state at level two is

$$|\varepsilon,\zeta,p\rangle = \varepsilon_{\mu\nu}\,\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|p\rangle + \zeta_{\mu}\alpha_{-2}^{\mu}|p\rangle$$

where $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$. Then,

$L_0 - a = 0 \Rightarrow A^2 = 4 - 2a > 0$, little gp is $SO(d)$

and we can take $\alpha_0^0 = A$ $\alpha_0^i = 0$.

$$L_1|\varepsilon,\zeta,p\rangle = (\alpha_2\cdot\alpha_{-1} + \alpha_1\cdot\alpha_0)|\varepsilon,\zeta,p\rangle$$

$$= \underbrace{(2A\varepsilon_{0\mu} + 2\zeta_{\mu})}_{\overset{!}{=}0}\underbrace{\alpha_{-1}^{\mu}|p\rangle}_{l.i.}$$

$$L_2|\varepsilon,\zeta,p\rangle = (\alpha_2\alpha_0 + \tfrac{1}{2}\alpha_1^2)|\varepsilon,\zeta,p\rangle$$

$$= \underbrace{(\varepsilon^{\mu}{}_{\mu} + 2A\zeta_0)}_{\overset{!}{=}0}|p\rangle$$

<u>Question</u>: under what condition is quadratic form

$$\mathcal{N} = \||\varepsilon,\zeta,p\rangle\|^2 \propto \varepsilon_{\mu\nu}\varepsilon^{\mu\nu} + \zeta_{\mu}\zeta^{\mu} = \varepsilon_{00}^2 - 2\varepsilon_{0i}^2 + \varepsilon_{ij}^2 - \zeta_0^2 + \zeta_i^2$$

(semi) pos. def. on sols of physical state conditions?

Observation: the "traceless part" of ε_{ij} does not appear in eqs. and contributes positively to $\mathcal{N}$.

$\rightsquigarrow$ assume $\varepsilon_{ij} = \varepsilon\,\delta_{ij}$

Then 1^{st} eq. says $\zeta_i = -A\varepsilon_{0i}$

$$\left.\begin{aligned} \zeta_0 &= -A\varepsilon_{00} \\ 2A\zeta_0 &= \varepsilon_{00} - \varepsilon d \end{aligned}\right\}$$

while 2^{nd}:

$\varepsilon_{00} = \varepsilon d + 2A\zeta_0$

$$\Rightarrow \quad \zeta_0 = -2A^2\zeta_0 - A d\,\varepsilon$$

$$\left[\begin{aligned} \zeta_0 &= -\frac{Ad}{1+2A^2}\,\varepsilon \\ \varepsilon_{00} &= \frac{d}{1+2A^2}\,\varepsilon \end{aligned}\right.$$

$$\cancel{\mathcal{N} = \frac{1}{(1+2A^2)^2}\Big[d^2\varepsilon^2 + d\varepsilon^2 - \nu\varepsilon}$$

$$\mathcal{N} = \frac{d^2\varepsilon^2}{(1+2A^2)^2} + d\varepsilon^2 - \frac{A^2 d^2}{(1+2A^2)^2}\varepsilon^2 + (A^2-2)\,\varepsilon_{0i}^2$$

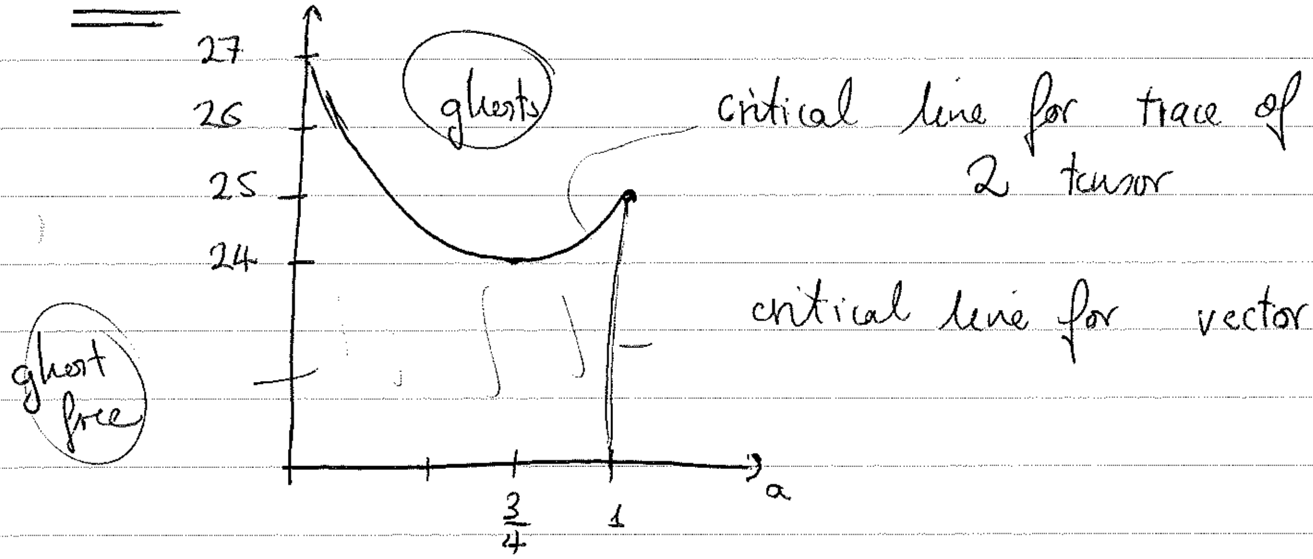
$$= \frac{d\varepsilon^2}{(1+2A^2)^2}\underbrace{\Big[(1+2A^2)^2 + d(1-A^2)\Big]}_{=\,(9-4a)^2 \,-\, d(3-2a)} + \underbrace{(A^2-2)}_{\cancel{4(a-1)}\;\; 2(1-a)}\varepsilon_{0i}^2$$

This is positive semi-definite iff

$$a \leq 1$$

$$d \leq \frac{(9-4a)^2}{3-2a}$$

$\underline{\underline{d = D-1}}$



$\to$ at $a=1$, the entire vector becomes "pure gauge"

at $d = \frac{(9-4a)^2}{3-2a}$, the scalar also becomes pure gauge

At $a=1$, $d=25$, the ~~extra~~ scalar null state is

$$\left(5\, {\alpha^0_{-1}}^2 + {\alpha^i_{-1}}^2 - 5A\, \alpha^0_{-2} \right) |p\rangle .$$

Because of

$$L_{-2}\,|p\rangle = \left(\alpha_{-2}\alpha_0 + \tfrac{1}{2}\alpha_{-1}\alpha_{-1} \right) |p\rangle$$

$$= \left(-A\,\alpha^0_{-2} - \tfrac{1}{2}{\alpha^0_{-1}}^2 + \tfrac{1}{2}{\alpha^i_{-1}}^2 \right) |p\rangle$$

and

$$L_{-1}^2 |p\rangle = \left(\alpha_{-2} \cdot \alpha_1 \quad \alpha_{-1} \cdot \alpha_0 + \left(\alpha_{-1} \cdot \alpha_0 \right)^2 \right) |p\rangle$$

$$= \left(\underset{\overset{\parallel}{2}}{A^2} \; {\alpha_{-1}^{0}}^2 - A \, \alpha_{-2}^{0} \right) |p\rangle$$

this can also be written as

$$\left(2 L_{-2} + 3 L_{-1}^2 \right) |p\rangle$$

showing that the state is indeed null, orthogonal to all physical states.

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Lemma: Say $|\psi\rangle$ obeys $L_0 |\psi\rangle = -1$ and $L_n |\psi\rangle = 0 \; \forall n > 0$ where L_n satisfy VO with $c = 26$.

Then

$$\left(L_{-2} + \frac{3}{2} L_{-1}^2 \right) |\psi\rangle$$

is physical ~~[illegible]~~, hence null.

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$$L_2 = -4 + \frac{8-2}{12} \cdot c + \frac{3}{2} \left(3 L_1 L_{-1} \right) |\psi\rangle$$

$$= -4 + 13 - 9 = 0$$

$$L_1 = 3 L_{-1} + \; + \frac{3}{2} (-3 - 3) = 0$$

$$\Rightarrow \| \cdot \|^2 = \cancel{-4 + 13} \; 0.$$ Follows from this

5 The celebrated no-ghost theorem

- $\mathcal{H}_{OCQ}$ is a pre-Hilbert space (i.e., inner product is pos. def.) if and only if,

either (1) $a=1$ and $D=26$

or (2) $a \le 1$ and $D \le 25$

- In case 1, $\tilde{\mathcal{F}}_{phys}$ has "unusual large number of null states", all of the form

$$\left(L_{-2} + \frac{3}{2} L_{-1}^2\right) |\chi\rangle$$

as in lemma. "critical string theory".

Full proof is not easy, proceeds by constructing explicit set of representatives in Light-cone gauge. We will give intuition for the statement.

"In case 1, gauge conditions + gauge invariance "remove 2 sets of oscillators"".

Based on this, one can expect (and prove) that theorem also holds if $\tilde{\mathcal{F}}$ is replaced with

$$\underbrace{\tilde{\mathcal{F}}^0 \otimes \tilde{\mathcal{F}}^1}_{} \otimes \mathcal{M}_{24}$$

Fock spaces for X^0, X^1 momenta and oscillators and $\mathcal{M}_{24}$ is "arbitrary" unitary Virasoro module of $c=24$. corresponding to "other string backgrounds".

Also, case ② can be understood to follow from ① by say tensoring

$$\mathcal{F}^0 \otimes \mathcal{H}_{24} \otimes \mathcal{F}^1$$

(and using 26-dim theory to remove $\tilde{\mathcal{F}}^1$ again) to understand "gauge conditions remove only light like oscillators", but there are far fewer null states.

- Roughly, case 2 arises from 1 by mistake, and $a=1$ $D=26$ is only truly relevant case in which string wants to live.

(Case 2 has problems at loop level, in some sense is really only a subsector of critical string.)

Lecture 8 Light-cone gauge

11/11/21

- OCQ is "straightforward and has many other insights to offer (see textbooks).
- Most importantly: covariant
- However: – spectrum not so easy to understand / grasp intuitively
  – not so easy to understand interactions.

Next week: cohomological resolution via BRST which does fit nicely with path-integrals ⇝ interactions.

Today: Light-cone gauge, which gives
- an alternative derivation of critical dimension
- a simple clear intuitive and easy-to-remember way of understanding the unitary string spectrum as being generated by transverse oscillators.

Basic idea: • Fix gauge and solve constraints completely, only quantise physical dofs.

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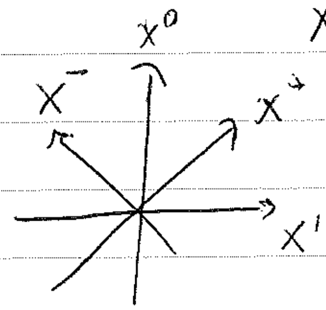
① Light-cone coordinates

- a class of linear, but non inertial coordinates on Minkowski spacetime $\mathbb{R}^{1,d}$, related to inertial coordinates via



$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^1) \qquad X^- = \frac{1}{\sqrt{2}}(X^0 - X^1)$$

- write X^I $I = 2, \ldots d$ for $d-1$ others unchanged
- note $\sqrt{2}$ in comparison with worldsheet $\sigma^\pm$.

- Metric is

$$-dx^{0\,2} + dx^{i\,2} = -2\,dx^+ dx^- + dx^{I\,2}$$

$$\begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

- This means: indices are raised and lowered with two sign changes; not a problem for X's but for quantities like D-momentum

$$p_+ = -p^- \qquad p_- = -p^+ \qquad p_I = p^I.$$

- Natural to view one of $X^\pm$ as "time". X^+: light-cone time. Then

$$p_+ = \frac{\partial}{\partial x^+} = -p^-$$

is "light-cone Hamiltonian".

- Light-cone velocity for massive particle $X^1 = \beta X^0$ is

$$\frac{X^-}{X^+} = \frac{1-\beta}{1+\beta} \in [0,\infty)$$

somewhat more like non-relativistic situation, however:

- We shall not quantize w.r.t. X^+. ("light-cone quantization").

Instead:

② Light-cone gauge

- a choice of parametrization for classical string motion that is like "static gauge ~~f~~ w.r.t. light-cone time", i.e.

$$X^+(\tau,\sigma) = p^+\alpha'\tau \quad (\text{closed string})$$
$$2p^+\alpha'\tau \quad (\text{open string})$$

· can be constructed e.g. from conformal gauge, that is preserved by any $\sigma^+ = f(\tilde\sigma^+)$ $\sigma^- = g(\tilde\sigma^-)$,

with the concrete choice $f = \tilde X^+_L$ $g = \tilde X^+_R$

indeed $f = \frac{1}{2} p^+\alpha' \tilde X^+_L$ ~~+~~ $\frac{1}{2} p^+\alpha' \tilde X^+_R = g$

so indeed $\qquad \tau = \frac{1}{2}(\sigma^+ + \sigma^-)$

$$X^+(\sigma) = \tilde X^+_L(\tilde\sigma^+) + \tilde X^+_R(\sigma^-) = p^+\alpha'\tau$$

- Alternatively, one can declare or argue that ~~the~~ the gauge is good for "physical string motion", and after the fact find σ-parametrization ~~such the~~ that gives conformal gauge; depends on starting point Polyakov or Nambu–Goto, but simpler than solving ~~a~~ Poisson equation. Details see Zwiebach.

- Certainly anyway, gauge is bad for $p^+ = 0$, which does occur for isolated strings. ~~certainly those with~~

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③ Solving the constraints

The constraints (N-G or P) in light-cone coordinates and gauge take the form (open string only from now on).

$$0 = X'_R \cdot X'_R = -2\, p^+ \alpha' X'^-_R(\sigma^-) + X'^I_R(\sigma^-)\, X'_{RI}(\sigma^-)$$

$$\left(= X'_L \cdot X'_L \right)$$

and contain the X'^-_R only linearly, so that we may solve:

$$X'^-_R(\sigma^-) = \frac{X'^I_R(\sigma^-)\, X'_{RI}(\sigma^-)}{2p^+\alpha'}$$

and recover $X^-(\tau,\sigma)$ up to an integration constant.

- At this point, the constraints have completely disappeared so that, among other things, solutions of classical string eqns obey a superposition principle for oscillations transverse (to light cone). But X^- is non-linear!

- In terms of oscillators, $X'_R = \sqrt{\frac{\alpha'}{2}} \sum \alpha_n e^{-in\sigma^-}$

$$X'^{I}_{R} X'_{R I} = \alpha' \sum e^{-in\sigma^-} L^{\perp}_{n}$$

where $L^{\perp}_{n} = \frac{1}{2} \sum_{\substack{I=2 \\ p\in\mathbb{Z}}}^{d} \alpha^{I}_{p} \alpha_{n-p\, I}$

"transverse Virasoro modes"

we ~~also~~ have

$$\sqrt{\frac{\alpha'}{2}}\, \alpha^-_n = \frac{1}{2p^+\alpha'} \alpha' L^{\perp}_{n}$$

$$\alpha^-_n = \frac{1}{\sqrt{2\alpha'}\, p^+} L^{\perp}_{n}$$

(recalling $\sqrt{2\alpha'}\, p^+ = \alpha^+_0$ this is nothing but

$L_n = 0$ when $\alpha^+_n = 0$ $\forall n \neq 0$.

with $\alpha_0^- = \sqrt{2\alpha'}\, p^-$

$$L_0^\perp = \alpha' p^I p_I + \underbrace{\sum |\alpha_p^\perp|^2}_{N^\perp}$$

we find upon remaining $n=0$ identity:

$$M^2 = 2p^+p^- - p^{\perp 2} = \frac{1}{\alpha'} N^\perp \geq 0$$

manifestly as advertised.

Basically, this will carry over to quantum theory, giving "no ghosts" for free.

(4) Caveats

It can be rewarding, however, before quantizing, to reflect on the interplay between this gauge fixing and the canonical formalism. Disclaimer: I don't understand this fully.

For illustration, <u>co</u> pt particle in proper time action

$$L = -m\sqrt{-\dot{x}^2}$$

Proceeding naively, we find

$$p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \quad ; \quad H = 0$$

While the latter (identity) is as expected from the canonical discussion in lecture 4 it leads to (possibly) puzzling statements like

$$\dot{x}^\mu_{po} = \{H, x^\mu\} = 0 \ ??$$

- Really however this is just a reflection of the fact that (evolution in) worldline time τ was fictitious and ambiguous to begin with
- Namely, the naive transition to the canonical formalism is not possible because the p_μ are not independent but satisfy the identity

$$p_\mu p^\mu = -m^2$$

As a ~~[illegible]~~ consequence, the map $(x^\mu, \dot{x}^\mu) \to (x^\mu, p_\mu)$ is not invertible, and (in this case) the Hamiltonian is "ambiguous". Picking e.g.

$$H = \frac{p_\mu p^\mu}{m^2} + 1$$

does not change "energy" on phase space but does induce a "correct" time evolution

$$\dot{x}^\mu = p^\mu / m^2$$

corresponding to proper time gauge

$$\sqrt{-\dot{x}^2} = \frac{1}{m}.$$

In this case, naive "Gupta-Bleuler" quantization

$$[p^\mu, x_\nu] = -i\,\delta^\mu{}_\nu$$

represented as usual on $L^2(\mathbb{R}^{1,d})$ gives after imposing the constraint $p^\mu p_\mu = -m^2$ as physical Hilbert space

$$\mathcal{H}_{\text{Phys}} = \{\Box\varphi = m^2\varphi\}$$ $\ni e^{-ip^0x^0 + ip^ix^i}$

e.g. with plane wave basis $e^{ip\cdot x}$ $p^2 = -m^2$

from which τ has completely disappeared, ~~although~~ while "physical time evolution" is generated by a Noether charge

$$p^0 = i\frac{\partial}{\partial X^0}$$

~~Imposing, light cone instead, log~~ (No null state b/c no weak constraints and a lot more to say)

Imposing, instead, light-cone gauge

$$X^+ = \frac{p^+\tau}{m^2} = -\frac{p_-\tau}{m^2}$$

first, the Lagrangian becomes

$$L = -\sqrt{2p^+\dot{x}^- - m^2\dot{x}^{i\,2}}$$

$$P_- = -\frac{p_-}{\ldots}$$

with $\dot{p}_- = -\frac{p^+}{\Gamma} = \frac{p_-}{\Gamma}$

$$\leadsto \Gamma = -L = 1$$

$$p_I = \frac{m^2 \dot{x}_I}{\Gamma} = m^2 \dot{x}_I$$

$$\Gamma = 1 \Rightarrow \dot{x}^- = -\frac{p_\perp^2}{m^2}\frac{1}{2p_-}$$

So we get a ~~and~~ mathematically fine phase space with d canonical pairs

$$\{p_I, x^I\} = \{p_-, x^-\} = 1$$

which because of

$$H = \frac{p_\perp^2}{2m^2} + 1$$

and other quirks is harder to interpret physically; there seems to be no time evolution for x^- (*), unclear relation to covariant wave-functions, Lorentz transformations are not all Noether charges etc.

(*) this actually is due to the fact that we used p_- in the gauge fixing(!). If instead we pick

$$x^+ = \lambda\tau$$

for some constant λ.

we find

$$H = -\frac{\lambda}{2p_-}\left(m^2 + p_\perp^2\right)$$

$$= -\lambda p_+ = -\lambda \partial_+$$

generating "light-cone time evolution", see ①

- The choice $\lambda = -\frac{p_-}{m^2}$ modifies "τ-dynamics" to a kind of world-line interaction picture in which "physical x^--observable" has explicit time dependence.

- For full exquisite discussion, including discussion/derivation of Lorentz-symmetry generators, see Zwiebach.

- The discussion for the string would need to be yet different yet, b/c we obtained canonical formalism and constraints not by gauge fixing N-G, but by gauging conformal symmetry à la Polyakov.

⑤ Quantization

In any event however, one finds that ~~gauge~~ ~~fixed~~ string in light-cone gauge has canonical variables

$$[p_I, x^J] = \delta_I^J \qquad I, J = 2, \ldots d$$

$$[p_-, x^-] = 1$$

and $D-2$ sets of oscillators

$$[\alpha_n^I, \alpha_{m\,J}] = i n \delta_J^I \delta_{n+m,0}$$

α_n^- (including $p^- = -p_+ = \frac{1}{\sqrt{2\alpha'}}\alpha_0^-$) are dependent ~~(canonical~~ variables

$$\alpha_n^- = -\frac{1}{\sqrt{2\alpha'}\,p_-} L_n^\perp$$

while α_n^+ oscillators (except $\alpha_-^+ \sim p^+$) have completely disappeared from the formalism.

- Based on reality $\alpha_n^{I\,\dagger} = \alpha_{-n}^I$, one constructs a manifestly pos. def. Fock space in the usual manner with only transverse oscillators excited. $\sim \mathcal{H}_{LCG}$

- The critical dimension and a now appear ~~somewhat~~ more directly but more handwavingly, & by inspection of mass formula

$$-p^2 = M^2 = \frac{1}{\alpha'}\left(N^\perp - a\right)$$

where $N^{\perp} = \sum_{p=1}^{\infty} \alpha_{-p} \alpha_{p}$

and a is "normal ordering constant" as before.

· Since at first excited level, there are only D-2 states, these can form good rep of a little gp only if little gp is $SO(D-2)$ i.e. $p^2=0$, i.e. $\underline{a=1}$

· Even more handwavingly, one observes that according Planck the g.s. energy is $\frac{1}{2}\omega$, so that

$$N^{\perp} = \sum_{p=1}^{\infty} \alpha_{-p}\alpha_{p} + \underbrace{\sum_{1}^{\infty} p \cdot \frac{(D-2)}{2}}_{= -\frac{D-2}{24}}$$

$= 1 \Leftrightarrow \underline{\underline{D=26}}$.

6 The crux

· ~~The decomp~~ Even accepting this, the decomposition of massive spectrum is complicated by the fact that (even if true little gp is $SO(D-1)$), even only $SO(D-2)$ is manifest, so we need some guesswork unless we invoke Lorentz trafos mixing
$$x^{I} \Leftrightarrow x^{\pm}$$

- These transformations do not respect light-cone gauge, so certainly they are not Noether symmetries after gauge fixing.

- However, by some procedure, one could verify that the expressions that <u>should</u> generate these transformations are

$$J^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\Big(\underbrace{x_0^I (L_0 \mp a) + (L_0 \mp a)\, x_0^I}_{\text{from } "p^- x_0^I"} \Big)$$

$$- \frac{i}{2\alpha' p^+} \sum_{n=1}^{\infty} \frac{1}{n}\left(L_{-n}\alpha_n^I - \alpha_{-n}^I L_n \right)$$

L_n satisfy Virasoro

$$[L_n, a_m^I] = -m\,\alpha^I_{n+m}$$

etc., long calculation reveals

$$[J^{-I}, J^{-J}] = -\frac{1}{\alpha' p^{+2}} \sum_n \left(\alpha_{-n}^I \alpha_n^J - \alpha_{-n}^J \alpha_n^I \right).$$

$$\left[\frac{1}{n}\left(\frac{D-2}{24} \mp a \right) + n\left(1 - \frac{D-2}{24} \right) \right]$$

$= 0 \iff D = 26$

$a = -1.$

<u>Upshot:</u> • It's complicated, but easy to remember

• Closed strings work similarly

• $\mathcal{H}_{\alpha Q} = \mathcal{H}_{LCG}$ ~~is~~ for $D=26$, $a=1$ is part of proof of no-ghost theorem.

Aside. Calculation of Lorentz-algebra relation $[M^{-I}, M^{-J}] = 0$ in bosonic string theory in light-cone gauge for calculation of critical dimension $D = 26$

Starting point is algebra of center of mass position and momenta, oscillators, and Virasoro modes.

$$[x_0^-, p^+] = -i \qquad [p^I, x_0^J] = -i\delta^{IJ}$$

$$\alpha_0^I = \sqrt{2\alpha'}\, p^I \qquad [\alpha_n^I, \alpha_m^J] = n\,\delta_{n+m,0}$$

$$L_0 = \frac{2\alpha'}{2} p^I p_I + \sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_n^I$$

$n \neq 0$
$$L_n = \frac{1}{2} \sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I$$

$$[L_n, x_0^I] = -i\sqrt{2\alpha'}\, \alpha_n^I$$

$$[L_0, x_0^I] = -i\, 2\alpha' p^I$$

$$[L_n, \alpha_m^I] = -m\, \alpha_{n+m}^I$$

$$[L_n, L_m] = (n-m) L_{n+m} + c_n\, \delta_{n+m,0}$$

$$c_n = \frac{D-2}{12}(n^3 - n)$$

$$[L_0, L_n] = -n L_n \qquad [L_0, \alpha_n^I] = -n\, \alpha_n^I$$

a: normal ordering constant

according Zwiebach (12.161)

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I (L_0 + a) + (L_0 + a) x_0^I \right)$$

$$- \frac{i}{\sqrt{2\alpha'}\, p^+} \sum_{n=1}^{\infty} \frac{1}{n} \left(L_{-n} \alpha_n^I - \alpha_{-n}^I L_n \right)$$

We wish to calculate for $I \neq J$ the commutator $[M^{-I}, M^{-J}]$. It consists of 6 types of terms (because M^{-I} contains 3 types of terms).

(I) $[x_0^- p^I, x_0^- p^J] = 0$

(II) $$\left[x_0^- p^I, -\frac{1}{4\alpha' p^+}\left(x_0^J (L_0 + a) + (L_0 + a) x_0^J \right) \right]$$

$$= \frac{-i}{4\alpha' p^{+2}} p^I \left(x_0^J (L_0 + a) + (L_0 + a) x_0^J \right)$$

(III) $$\left[x_0^- p^I, - \frac{i}{\sqrt{2\alpha'}\, p^+} \sum_{n=1}^{\infty} \frac{1}{n} \left(L_{-n} \alpha_n^J - \alpha_{-n}^J L_n \right) \right]$$

$$= \frac{1}{\sqrt{2\alpha'}\, p^{+2}} \sum_{n=1}^{\infty} \frac{1}{n} \left(L_{-n} \alpha_n^J - \alpha_{-n}^J L_n \right) p^I$$

we also have same term with $I \leftrightarrow J$ to antisymmetrize.

IV $\left[\frac{1}{4\alpha' p^+} \left(x_0^I (L_0+a) + (L_0+a) x_0^I \right), \frac{1}{4\alpha' p^+} \left(x_0^J (L_0+a) + (L_0+a) x_0^J \right) \right]$

$$= \frac{1}{(4\alpha' p^+)^2} (-i\,2\alpha') \cdot \Big[x_0^I p^J (L_0+a) - x_0^J p^I (L_0+a)$$
$$+ (L_0+a) x_0^I p^J - (L_0+a) p^I x_0^J$$
$$+ p^J x_0^I (L_0+a) - x_0^J p^I (L_0+a)$$
$$+ (L_0+a) p^J x_0^I - (L_0+a) p^I x_0^J \Big]$$

$$= -\frac{i}{4\alpha' p^{+2}} \Big[(x_0^I p^J - x_0^J p^I)(L_0+a)$$
$$+ (L_0+a)(x_0^I p^J - x_0^J p^I) \Big]$$

This cancels against II and $I \leftrightarrow J$:

$$-\frac{i}{4\alpha' p^{+2}} \Big[(p^I x_0^J - p^J x_0^I)(L_0+a)$$
$$+ (L_0+a)(p^I x_0^J - p^J x_0^I) \Big]$$

(V) $$\left[\frac{1}{4\alpha' p^+} \left(x_0^I (L_0+a) + (L_0+a) x_0^I \right), \frac{i}{\sqrt{2\alpha'} p^+} \sum_{n=1}^{\infty} \frac{1}{n} \left(L_{-n} \alpha_n^J - \alpha_{-n}^J L_n \right) \right]$$

(we here use that $[L_0, L_{-n}\alpha_n^I] = 0$ to only pick terms for $[x_0, L_n]$.)

$$= \frac{1}{4\alpha' p^{+2}} (L_0+a) \sum_{n=1}^{\infty} \frac{1}{n} \Big(-\alpha_{-n}^I \alpha_n^J + \alpha_{-n}^J \alpha_n^I$$
$$- \alpha_{-n}^I \alpha_n^J + \alpha_{-n}^J \alpha_n^I \Big)$$

$$= \frac{1}{2\alpha' p^{+2}} (L_0+a) \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^J \alpha_n^I - \alpha_{-n}^I \alpha_n^J \right)$$

this is already anti-symmetric in I, J, but we're still getting the same term again, for total of

$$\frac{1}{\alpha' p^{+2}} (L_0+a) \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^J \alpha_n^I - \alpha_{-n}^I \alpha_n^J \right)$$

Final term is most complicated:

(iii) $$-\frac{1}{2\alpha' p^{+2}} \sum_{n,m} \frac{1}{nm} \left[\left(L_{-n}\alpha^I_n - \alpha^I_{-n} L_n \right),\ L_{-m}\alpha^J_m - \alpha^J_{-m} L_m \right]$$

$$= -\frac{1}{2\alpha' p^{+2}} \sum_{n,m} \frac{1}{nm}$$

$$\Big[(m-n) L_{-m-n} \alpha^I_n \alpha^J_m + L_{-n} n \alpha^I_{n-m} \alpha^J_m - L_{-m} m \alpha^J_{m-n} \alpha^I_n$$

$$- m \alpha^J_{-m-n} \alpha^I_n L_m - n \alpha^J_{-m} L_{-n} \alpha^I_{n+m} - \alpha^J_{-m} \left((-n-m) L_{m-n} + c_{-n} \delta_{nm} \right) \alpha^I_n$$

$$+ n \alpha^I_{-n-m} L_n \alpha^J_m - \alpha^I_{-n} \left((n+m) L_{n-m} + c_n \delta_{nm} \right) \alpha^J_m + L_{-m} \alpha^J_{-n} m \alpha^J_{m+n}$$

$$+ \alpha^I_{-n} m \alpha^J_{n-m} L_m + \alpha^J_{-m} (-n) \alpha^I_{m-n} L_n + \alpha^I_{-n} \alpha^J_{-m} (n-m) L_{n+m} \Big]$$

First, the terms involving c_n:

$$-\frac{1}{2\alpha' p^{+2}} \sum_n \frac{1}{n^2} \left(\alpha^J_{-n} \alpha^I_n c_n - \alpha^I_{-n} \alpha^J_n c_n \right)$$

$$\cancel{\frac{1}{\alpha' p^{+2}} \sum_n \alpha^I_{-n} \alpha^J_n \frac{1}{n} \frac{D-2}{24}} -$$

$$= -\frac{1}{\alpha' p^{+2}} \sum_n \left(\alpha^I_{-n} \alpha^J_n - \alpha^J_{-n} \alpha^I_n \right) \left(\frac{1}{n} \frac{D-2}{24} - n \frac{D-2}{24} \right)$$

Now collect terms with negative Virasoro modes, or positive, together

$$\left(-\frac{1}{2\alpha' p^{+2}}\right) \sum_{k=1}^{\infty} L_{-k} \left(\sum_{m=1}^{k-1} \left(\frac{1}{m} - \frac{1}{k-m}\right) \alpha^I_m \alpha^J_{k-m} \right.$$

$$+ \sum_{m=1}^{\infty} \frac{1}{m} \alpha^I_{k-m} \alpha^J_m - \sum_{m=1} \frac{1}{m} \alpha^J_{k-m} \alpha^I_m$$

$$\left. - \sum_{m=1}^{\infty} \frac{1}{m} \alpha^J_{-m} \alpha^I_{k+m} + \sum \frac{1}{m} \alpha^I_{-m} \alpha^J_{k+m} \right)$$

$$+ \sum_{n,m} \alpha^J_{-m-n} \alpha^I_{n+m}$$

$$+ \sum_{n>m} \frac{1}{nm} \alpha^J_{-m} (n+m) L_{m-n} \alpha^I_n$$

$$+ \sum_{m>n} \frac{-1}{nm} \alpha^I_{-n} (n+m) L_{n-m} \alpha^J_m$$

from $n=m$

$$+ \sum \frac{1}{n^2} \left(\alpha^J_{-n} 2n L_0 \alpha^I_n - \alpha^I_{-n} 2n L_0 \alpha^J_n \right)$$

$$+ \sum_{k=1}^{\infty} \left(\sum_{m=1}^{k-1} \left(\frac{1}{m} - \frac{1}{k-m}\right) \alpha^I_{m-k} \alpha^J_{-m} \right.$$

$$+ \sum_{m=1}^{\infty} \frac{1}{m} \alpha^I_{-m} \alpha^J_{m+k} - \sum_{m=1}^{\infty} \frac{1}{m} \alpha^J_{-m} \alpha^I_{m-k}$$

$$\left. + \sum_{m=1}^{\infty} \frac{1}{m} \alpha^I_{-m-k} \alpha^J_m - \sum_{m=1}^{\infty} \frac{1}{m} \alpha^J_{-m-k} \alpha^I_m \right) L_k$$

$$- \sum_{n,m} \alpha^I_{-n-m} \alpha^J_{n+m}$$

$$+ \sum_{m>n} \frac{1}{nm} \alpha^J_{-m} (n+m) L_{m-n} \alpha^I_n - \sum_{n>m} \frac{1}{nm} \alpha^I_{-n} (n+m) L_{n-m} \alpha^J_m$$

$$
\begin{aligned}
-\frac{1}{2\alpha' p^+}\Bigg[\sum_{k=1}^{\infty} L_{-k} \Bigg(& \sum_{m=1}^{k-1} \frac{1}{m}\left(\alpha^I_m \alpha^J_{k-m} - \alpha^J_m \alpha^I_{k-m}\right) \\
& - \sum_{m=1}^{\infty} \frac{1}{m}\left(\alpha^I_m \alpha^J_{k-m} - \alpha^J_m \alpha^I_{k-m}\right) \\
& + \sum_{m=1}^{\infty} \frac{1}{m}\left(\alpha^I_{-m} \alpha^J_{k+m} - \alpha^J_{-m} \alpha^I_{k+m}\right) \\
& + \sum_{m=1}^{\infty} \left(\frac{1}{m} + \frac{1}{k+m}\right) \left(\alpha^I_{k+m} \alpha^J_{-m} - \alpha^J_{k+m} \alpha^I_{-m}\right) \Bigg) \\
+ \sum_{k=1}^{\infty} \Bigg(& \sum_{m=1}^{k-1} \frac{1}{m}\left(\alpha^I_{m-k} \alpha^J_{-m} - \alpha^J_{m-k} \alpha^I_{-m}\right) \\
& + \sum_{m=1}^{\infty} \frac{1}{m}\left(\alpha^I_{-m} \alpha^J_{m-k} - \alpha^J_{-m} \alpha^I_{m-k}\right) \\
& + \sum_{m=1}^{\infty} \frac{1}{m}\left(\alpha^I_{-m-k} \alpha^J_{m} - \alpha^J_{-m-k} \alpha^I_{m}\right) \\
& + \sum \left(\frac{1}{m} + \frac{1}{m+k}\right) \left(\alpha^J_{-m-k} \alpha^I_{m} - \alpha^I_{-m-k} \alpha^J_{m}\right) \Bigg) L_k \\
& + \sum_{n,m} \alpha^J_{-m-n} \alpha^I_{n+m} - \sum_{n,m} \alpha^I_{-n-m} \alpha^J_{n+m} \\
& - \sum_{n>m} \frac{1}{n} (n+m) \left(\alpha^J_{-n} \alpha^I_{n} - \alpha^I_{-n} \alpha^J_{n}\right) \\
& - \sum_{m>n} \frac{1}{m} (n+m) \left(\alpha^J_{-m} \alpha^I_{m} - \alpha^I_{-m} \alpha^J_{m}\right) \\
& + \sum_{n=1}^{\infty} \frac{2}{n} \left(\alpha^J_{-n} \alpha^I_{n} - \alpha^I_{-n} \alpha^J_{n}\right) L_0 \\
& - \sum_{n=1}^{\infty} 2 \left(\alpha^J_{-n} \alpha^I_{n} - \alpha^I_{-n} \alpha^J_{n}\right) \Bigg]
\end{aligned}
$$

8

$$-\frac{1}{2\alpha' p^{+2}}\left[\sum_{k=1}^{\infty} L_{-k}\frac{-1}{k}\left(\alpha_k^I\alpha_0^J-\alpha_k^J\alpha_0^I\right)\right.$$

$$\left.-\sum_{k=1}^{\infty}\frac{1}{k}\left(\alpha_{-k}^I\alpha_0^J-\alpha_{-k}^J\alpha_0^I\right)L_k\right]$$ this cancels II

$$-\frac{1}{\alpha' p^{+2}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\alpha_{-k}^J\alpha_k^I-\alpha_{-k}^I\alpha_k^J\right)L_0$$ this cancels part of V

$$-\frac{1}{2\alpha' p^{+2}}\left(\sum_{n,m}\left(\alpha_{-m-n}^J\alpha_{m+n}^I-\alpha_{-n-m}^I\alpha_{n+m}^J\right)\right.$$

$$-2\sum_n\left(\alpha_{-n}^J\alpha_n^I-\alpha_{-n}^I\alpha_n^J\right)$$

$$\left.-2\sum_n{}'\frac{1}{n}\sum_{m=1}^{n-1}(n+m)\left(\alpha_{-n}^J\alpha_n^I-\alpha_{-n}^I\alpha_n^J\right)\right)$$

$$=-\frac{1}{2\alpha' p^{+2}}\sum_k\left(\alpha_{-k}^J\alpha_k^I-\alpha_{-k}^I\alpha_k^J\right)\underbrace{\left(\cancel{(k-1)-2-\frac{2}{k}\left(k-1+\frac{k(k-1)}{2}\right)}\right)}_{\cancel{\text{part}}}$$

$$\cancel{k+1-\frac{2}{k}(k-1)\frac{k+2}{2}}$$

$$\cancel{=\frac{1}{k}\left(k^2+k-k^2+k-2\right)}$$

$$\left((k-1)-2-2(k-1)-2\frac{1}{k}\frac{k(k-1)}{2}\right)$$

$$=(k-1)-2k-(k-1)=-2k$$

pulling together,

$$-\frac{1}{\alpha' p^{+2}} \sum_n \left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right)\left(\frac{1}{n}\frac{D-2}{24} - n\frac{D-2}{24}\right)$$

$$+\frac{1}{\alpha' p^{+2}} \sum_n \left(\alpha^J_{-n}\alpha^I_n - \alpha^I_{-n}\alpha^J_n\right)\left(\frac{1}{n}\cdot a\right)$$

$$+\frac{1}{\alpha' p^{+2}} \sum_n \left(\alpha^J_{-n}\alpha^I_n - \alpha^I_{-n}\alpha^J_n\right) n$$

$$= -\frac{1}{\alpha' p^{+2}} \sum_n \left(\alpha^I_{-n}\alpha^J_n - \alpha^J_{-n}\alpha^I_n\right)\left(\frac{1}{n}\left(\frac{D-2}{24} + a\right) + n\left(1 - \frac{D-2}{24}\right)\right)$$

exactly as Zwiebach claims

For this to vanish we need $\frac{D-2}{24} = 1$ and

$a = -1$.