Lecture 23 Worldsheet supersymmetry

- In the previous lecture, we indicated why a supersymmetric extension of the bosonic string might help to
  - produce spacetime fermions
  - stabilize the vacuum
- For the process, we will utilize a number of (by now) familiar constructions (holomorphic/chiral splitting of the worldsheet, topological sectors through boundary conditions, orbifolding) while respecting our holy principles (conformal invariance, operator state correspondence, modular invariance)
- The first important novelty, which thereby is the defining feature of the "operating system" superstring, is a modification of the gauge algebra accompanied by various (super-) geometric considerations, most of which we will however postpone to future installment of the course.

Warning: Supersymmetry is a notorious "cauchemar de signes", which are so convention dependent and compound with factors of i that it is not easy to find a section of consistent choice (let alone a unique one). I won't try.

Example If θ_1, θ_2 are complex Grassman variables (or other odd quantities)

$$\overline{\theta_1\theta_2} = \bar{\theta}_2\bar{\theta}_1 = -\bar{\theta}_1\bar{\theta}_2 \quad \text{(physics convention)}$$

$$= -\bar{\theta}_2\bar{\theta}_1 = \bar{\theta}_1\bar{\theta}_2 \quad \text{(math convention)}$$

[The alphabet is still in conflict]

(in physics convention, if θ_1, θ_2 are real, then $\theta_1\theta_2$ is imaginary, how crazy is that!).

① Super Virasoro algebra

- The physical Hilbert space of the bosonic string is the result of gauging conformal symmetry of the worldsheet theory.
- Conformal symmetry itself is an extension of affine group, which in euclidean signature and complex coordinates is generated after adjoining dilatations by vector fields

$$\ell_{-1} = -\partial_z \,, \quad \ell_0 = -z\partial_z \quad \Big| \quad \tilde{\ell}_{-1} = -\partial_{\bar{z}} \,; \; \tilde{\ell}_0 = -\bar{z}\partial_{\bar{z}}$$

[carry along for the ride

- first by special conformal transformations

$$\ell_1 = -z^2\partial_z \qquad \text{(which continue globally to } \mathbb{P}^1\text{)}$$

- then (locally) by ∞ more ~~local~~ vector fields

$$\ell_n = -z^{n+1}\partial_z$$

$$[\ell_n, \ell_m] = (n-m)\,\ell_{n+m}$$

Note: everything is graded by conformal weight = eigenvalue of ℓ_0.

$h(l_n) = -n$

. h(state in Virasoro rep) = highest weight + level

weight of associated conformal primary ϕ with mode expansion

$$\phi(z) = \sum_m \phi_m z^{-m-h}$$

$$[L_n, \phi_m] = ((h-1)n - m)\,\phi_{n+m}$$

e.g. $h(T(z)) = 2$

• Now, the first thing to add in a supersymmetric extension is a squareroot of translations, represented geometrically by an odd vectorfield. In general spacetime dimension, this requires complicated discussion of spinors (and we'll have to do it eventually). Because the 2-dim. rotation gp is abelian, any (complex) irrep is one-dimensional, so it suffices to add single odd (complex) variable Θ, focus on the grading and take the usual

$$g_{-\frac{1}{2}} = \frac{\partial}{\partial\Theta} - \Theta\frac{\partial}{\partial z}$$

$$g_{-\frac{1}{2}}^2 = -\partial_z = l_{-1}, \quad [g_{-\frac{1}{2}}, g_{-\frac{1}{2}}] = 2\,l_{-1}$$

to repair the grading$_{\text{spin}}$, change l_0 to

$$\cancel{-z\partial_z + \frac{1}{2}\Theta\partial_\Theta} \qquad l_0 = -z\partial_z - \frac{1}{2}\Theta\partial_\Theta$$

s.t. $h(\Theta) = -\frac{1}{2}$ $\quad h(\partial_\Theta) = \frac{1}{2}$

the special superconformal generators can be intuited ~~to~~ from the reflection

$$z = \frac{1}{w} \qquad \Theta = \frac{1}{w}\eta \qquad (\Theta \in \mathcal{O}_{\mathbb{P}^1}(-1))$$

$$\partial_z = -w^2\partial_w - w\eta\partial_\eta$$

$$\partial_\Theta = w\partial_\eta \quad \rightsquigarrow \quad \partial_\Theta - \Theta\partial_z = w\partial_\eta + w\eta\partial_w$$

after adjusting some signs to be

$$g_{1/2} = z(\partial_\Theta - \Theta\partial_z) \qquad (\{g_{1/2}, g_{-1/2}\} = -2z\partial_z - \Theta\partial_\Theta = 2l_0$$

$$l_1 = -z^2\partial_z - z\Theta\partial_\Theta = g_{1/2}^2.$$

and the rest by "educated guess"}

$$l_n = -z^{n+1}\partial_z - \frac{n+1}{2}z^n\Theta\partial_\Theta \qquad (n \in \mathbb{Z})$$

$$g_r = z^{r+\frac{1}{2}}\left(\partial_\Theta - \Theta\partial_z\right) \qquad (r \in \mathbb{Z} + \tfrac{1}{2}).$$

$$[l_n, l_m] = (n-m)\, l_{n+m}$$

$$[l_n, g_r] = \left(\frac{n}{2} - r\right) g_{n+r}$$

$$\{g_r, g_s\} = 2\, l_{r+s}$$

This ($\mathcal{N}=1$, unextended) super-Virasoro algebra can more invariantly be characterized by "invariance of super-conformal structure" which however at the moment I do not want to define. As a shortcut, observe that conformal (i.e. holomorphic + anti-holomorphic) vectorfields can be characterized by the conditions

$$[\xi, \partial_z] \propto \partial_z \qquad [\xi, \bar{\partial}_{\bar{z}}] \propto \bar{\partial}_{\bar{z}}$$

$$\Rightarrow \xi = a\partial_z + b\partial_{\bar{z}} \; ; \; \bar{\partial} a = \partial b = 0.$$

recall that (in opposition to $g_{1/2}$, $\tilde{g}_{-1/2}$), we have "super covariant derivative"

$$\mathcal{D} = \partial_\theta + \theta\partial_z \qquad \bar{\mathcal{D}} = \bar{\partial}_{\bar{\theta}} + \bar{\theta}\bar{\partial}_{\bar{z}}$$

$$\{\mathcal{D}, g_{-\frac{1}{2}}\} = 0 \qquad \mathcal{D}^2 = \partial_z$$

and state that superconformal vectorfields are precisely those $\xi_\eta = a\partial_z + b\partial_{\bar{z}} + c\partial_\theta + d\partial_{\bar{\theta}}$ (even or odd) s.t.

$$[\xi_\eta, \mathcal{D}] \propto \mathcal{D}$$

$$[\xi_\eta, \bar{\mathcal{D}}] \propto \bar{\mathcal{D}}$$

[graded commutator.

<u>Exercise</u>: Show that this recovers super Virasoro.

② Superconformal field theory

- We now seek variational principles which upon quantization will realize the super Virasoro by operators L_n, G_r on their Hilbert space, possibly up to a central extension. From their commutation with L_n's we anticipate that the G_r will assemble into a weight $\frac{3}{2}$ (conformal) primary

$$G(z) = \sum G_r z^{-r-\frac{3}{2}}$$

in euclidean signature on the plane. For now, we work in Minkowski signature, but light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$.

- For some reason, such variational principles can be constructed systematically by coupling super matter to 2d supergravity in superspace (and then gauge fixing)
- We will proceed in a more pedestrian way, at the expense of not utilizing θ, ∂_θ, but only super vector fields in fields space.

A natural starting point is to add conformal fermions to our free boson.

$$S = \frac{1}{2\pi} \int_\Sigma d^2\sigma \left(\partial_+ X \partial_- X + i \psi_+ \partial_- \psi_+ \right)$$

where: $\cdot$ $\alpha' = 1$ (otherwise, add $\frac{1}{\alpha'}$ to bosons but not usually to fermions)

· $\psi_+(\tau,\sigma)$ is (real) anticommuting lest the action be a total derivative ($\psi_+\partial_-\psi_+ = -\partial_-\psi_+\psi_+ = \psi_+\partial_-\psi_+$), and such that equation of motion is non trivial

$$\delta S = \int_\Sigma (\delta\psi_+ \partial_-\psi_+ + \psi_+\partial_-\delta\psi_+)$$

$$= \int_\Sigma \underbrace{\delta\psi_+\partial_-\psi_+ - \partial_-\psi_+\delta\psi_+}_{2\,\delta\psi_+\partial_-\psi_+} + \int_{\partial\Sigma}\psi_+\delta\psi_+$$

Sol'n: $\psi_+ = \psi_+(\sigma^+) =$

$\rightsquigarrow$ $\partial_-\psi_+ = 0$ upholds if $\int_{\partial\Sigma}\psi_+\delta\psi_+ = 0$

Traditionally, one pauses here to observe that

- for closed strings on cylinder $\Sigma = \mathbb{R}\times S^1$, this allows choice between

· periodic boundary conditions $\psi_+(\tau,\sigma+2\pi) = \psi_+(\tau,\sigma)$

(Ramond)

· anti-periodic boundary conditions $\psi_+(\tau,\sigma+2\pi) = -\psi_+(\tau,\sigma)$

(Neveu-Schwarz)

with similar looking mode expansion

$$\psi_+ = \psi_+(\sigma^+) = \sum \tilde{b}_r\, e^{-ir\sigma^+}$$

but $r\in\mathbb{Z}$ Ramond $r\in\mathbb{Z}+\frac{1}{2}$ Neveu-Schwarz

- The i makes action real as mentioned above
- ψ_+ is a conformal field of weight $(0, \frac{1}{2})$, which notation is unfortunate but conventional.
- (Both parts of) S' are "manifestly" invariant under conformal transformations generated by

$$\zeta = \zeta^+ \partial_+ + \zeta^- \partial_-$$

$$\delta X = \zeta^+ \partial_+ X + \zeta^- \partial_- X$$

$$\delta \psi_+ = \zeta^+ \partial_+ \psi_+ + \zeta^- \partial_- \psi_+ + \frac{1}{2} \partial_+ \zeta^+ \psi_+$$

e.g. $\zeta^- = 0$, fermionic piece

$$\delta S' \supset \int \zeta^+ \partial_+ \psi_+ \partial_- \psi_+ + \frac{1}{2} \partial_+ \zeta^+ \psi_+ \partial_- \psi_+$$

$$+ \underbrace{\psi_+ \partial_- \zeta^+ \partial_+ \psi + \frac{1}{2} \psi_+ \partial_- (\partial_+ \zeta^+ \psi_+)}$$

$$= \psi_+ \zeta^+ \partial_+ \partial_- \psi_+ + \frac{1}{2} \partial_+ \zeta^+ \psi_+ \partial_- \psi_+$$

$$= \int \partial_+ (\zeta^+ \partial \psi_+ \partial_- \psi_+) \, d^2\sigma = 0.$$

Noether currents: $\zeta^+ T_{++} + \zeta^- T_{--}$

where $T_{++} = \partial_+ X \partial_+ X + \frac{1}{2} i \psi_+ \partial_+ \psi_+$

$T_{--} = \partial_- X \partial_- X$.

are by now familiar expressions for stress energy tensor.

The novel thing is that taken together, S' is invariant under the odd symmetry parametrized by odd "function" η^+ (half-vector) with $\partial_- \eta^+ = 0$ and

$$\delta X = -i\eta^+ \psi_+ \qquad \delta\psi_+ = \eta^+ \partial_+ X$$

Namely, $\delta\left(\partial_+ X \partial_- X + i \psi_+ \partial_- \psi_+ \right)$

$$= -i\, \partial_+(\eta^+\psi_+)\, \partial_- X - i \partial_+ X \underbrace{\partial_-(\eta^+\psi_+) + i\eta^+ \partial_+ X \partial_- \psi_+}_{=0 \text{ b/c } \partial_-\eta^+ = 0} + i\psi_+ \partial_-(\eta^+ \partial_+ X)$$

$$= -i\, \partial_+(\eta^+\psi_+)\, \partial_- X - i\, \eta^+\psi_+\, \partial_+ \partial_- X$$

$$= -i\, \partial_+\left(\eta^+\psi_+ \partial_- X\right) \quad \rightsquigarrow \quad \delta S' = 0.$$

The Noether current is $\eta^+ G_{++}$ where G_{++} is the anticipated weight $(0, \frac{3}{2})$ field

$$G_{++} = \sqrt{2}\, \partial_+ X \psi_+ .$$

③ Sectors and modes

- The theory we just described has so-called $\mathcal{N}=(0,1)$ superconformal symmetry.

- Adding $i\psi_-\partial_+\psi_-$ would lead to $\mathcal{N}=(1,1)$ and we could write action for 2d Majorana fermion $\binom{\psi_+}{\psi_-}$ using 2d γ-matrices à la Dirac.

- For the closed string this is not necessary but if we did we could choose between Ramond and Neveu-Schwarz on the left and right independently.

Also note:

- η^+ and hence G_{++} have same boundary conditions as ψ_+. (ditto for ψ_-)

- Therefore, to preserve one common superconformal susy, if we have several ψ_+^μ (in correspondence with X^μ's) their boundary conditions have to be aligned. (ditto but independently for ψ_-)

- This alignment is also required for spacetime Lorentz invariance

- On the other hand, spacetime translations can only act on X^μ's, not on ψ^μ. ~~consistent with~~ This is consistent with both R and NS and implies that Poincaré invariance is preserved in all sectors.

Finally, after Wick rotation to euclidean signature and conformal map from cylinder to plane via $z = e^{i\sigma^-}$, the bosons have the familiar mode expansion "under the path-integral"

$$\partial X = i\sqrt{\frac{1}{2}} \sum \alpha_n z^{-n-1} \qquad = \left(\frac{\partial\sigma^-}{\partial z}\right) \partial_- X$$

$$\bar\partial X = i\sqrt{\frac{1}{2}} \sum \tilde\alpha_n \bar z^{-n-1}$$

the fermions

$$\psi = \left(\frac{\partial\sigma^-}{\partial z}\right)^{1/2} \psi_-(\sigma^-(z)) = (i)^{1/2} \sum b_r z^{-r-\frac{1}{2}}$$

$$\bar\psi = \left(\frac{\partial\sigma^+}{\partial\bar z}\right)^{1/2} \psi_+(\sigma^+(\bar z)) = i^{1/2} \sum \tilde b_r \bar z^{-r-\frac{1}{2}}$$

are single-valued in NS sector, but have a branch cut in Ramond sector.

Thus implies: Neveu-Schwarz b.c. are the "standard" from path-integral point of view, vertex operators can be constructed from ψ variables

- Ramond boundary conditions are best thought of as twisted sector w.r.t. worldsheet fermion number (which acts by (-1) on $\psi_\pm$ and $+1$ on X).
- corresponding vertex operators can not be written in terms of worldsheet fields, but rather involve b.c. in path-integral. Which is unfortunate for calculations.

- For the open string, we recall that both Dirichlet

$$\partial_+ X = -\partial_- X \quad \text{at } \partial\Sigma$$

and Neumann $\partial_+ X = \partial_- X$

are compatible with conformal invariance, in the sense that

$$\zeta^+ T_{++} = \zeta^- T_{--} \quad \text{at } \partial\Sigma$$

so that of current $j_0 = j_+ - j_-$ does not flow off the boundary preserved by vector field $\zeta^+ = \zeta^-$ at $\partial\Sigma$

. Then observe that we need both ψ_+ and ψ_- lest

$\psi_+ \delta\psi_+ |_{\partial\Sigma} = 0$ implies $\psi_+ = \text{const.}\ \forall\, \sigma^+$, eliminates completely.

. Then by consideration of $\eta^+ G_{++} = \eta^- G_{\mp -}$ we find that b.c. compatible with $\mathcal{N}=1$ superconformal preserving super boundary are

Neuman-Neuman on X

$$\psi_+ = \psi_- \quad \text{at} \quad \sigma = 0$$

$$\psi_+ = \psi_- \quad \text{at} \quad \sigma = \pi \qquad \text{(Ramond)}$$

$$\psi_+ = -\psi_- \quad \text{at} \quad \sigma = \pi \qquad \text{(Neveu-Schwarz)}$$

Dirichlet-Dirichlet on X

$$\psi_+ = -\psi_- \quad \text{at} \quad \sigma = 0$$

$$\psi_+ = -\psi_- \quad \text{at} \quad \sigma = \pi \qquad \text{(Ramond)}$$

$$\psi_+ = \psi_- \quad \text{at} \quad \sigma = \pi \qquad \text{(Neveu-Schwarz)}$$

Dirichlet-Neumann on X

$$\psi_+ = \psi_- \quad \text{at} \quad \sigma = 0$$

$$\psi_+ = -\psi_- \quad \text{at} \quad \sigma = \pi \qquad \text{(Ramond)}$$

$$\psi_+ = \psi_- \quad \text{at} \quad \sigma = \pi \qquad \text{(Neveu-Schwarz)}$$

Upshot: $\psi_- = \sum b_r e^{-ir\sigma^-}$ and $\psi_+ = \sum \tilde{b}_r e^{-ir\sigma^+}$

are related by $\tilde{b}_r = \pm b_r$ and r has same moding as X in Ramond sector and opposite in Neveu-Schwarz.

(Check this page again !!!)

Lecture 24 Spacetime Supersymmetry

(most likely, we'll get about to spacetime fermions)

Last time: · Adding worldsheet fermions $\psi_\pm$ to boson X with action

$$S = \frac{1}{2\pi}\int d^2\sigma \left(\partial_+ X \partial_- X + i\psi_+\partial_-\psi_+ + i\,\psi_-\partial_+\psi_-\right)$$

gives worldsheet theory with $\mathcal{N}=(0,1)$ (or $\mathcal{N}=(1,1)$) superconformal invariance generated by

$$T_{\pm\pm} = \partial_\pm X\, \partial_\pm X + \frac{1}{2}\, i\psi_\pm \partial_\pm \psi_\pm$$

$$G_{\pm\pm} = \sqrt{2}\,\partial_\pm X\, \psi_\pm$$

- Anticipating the string construction, spacetime Poincaré invariance is compatible with two basic choices of periodicity / boundary conditions on the ψ's called Ramond (R) and Neveu-Schwarz (NS) sectors.

Today: · build a superstring in $(D=d+1)$-dimensional Minkowski space based on worldsheet variables

$$\left(X^\mu, \psi^\mu_\pm\right)_{\mu=0,\dots,d}$$

· quantize under the superconformal constraints $T=G=0$

· study the spectrum.

Upshot: · critical dimension $D=10$

· there are various ways to combine left + right, open + closed sectors, some of which have a very pleasing spacetime spectrum (spacetime supersymmetric)

① Quantization of oscillators

· The canonical commutation relations for the ① right-moving modes,

$$\partial_- X^\mu = \sqrt{\frac{1}{2}} \sum_n \alpha_n^\mu e^{-in\sigma^-} \qquad n \in \mathbb{Z}$$

$$\psi_-^\mu = \sum b_r^\mu e^{-ir\sigma^-} \qquad r \in \mathbb{Z} + \tfrac{1}{2} \quad NS$$

$$r \in \mathbb{Z} \quad R$$

are

$$[\alpha_n^\mu, \alpha_m^\nu] = \eta^{\mu\nu} n\, \delta_{n+m,0}$$

(supplemented with $[p^\mu, x^\nu] = -i\eta^{\mu\nu}$ where $\alpha_0^\mu \sim p^\mu$ is spacetime mom. and x_0^μ com position)

$$\{ b_r^\mu, b_s^\nu \} = \eta^{\mu\nu} \delta_{r+s,0}$$

(No zero modes for NS
Clifford algebra for R)
↪ spacetime fermions.

· Under the reality condition

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$$

$$(b_r^\mu)^\dagger = b_{-r}^\mu$$

we can build an (indefinite) bosonic/fermionic Fock space on a vacuum $|\Omega\rangle$ representing the zero modes in the various sectors (more momentarily) and annihilated by $\alpha^\mu_n |\Omega\rangle = 0 \quad n>0 \qquad b^\mu_r |\Omega\rangle = 0 \quad r>0$.

Irrespective of boundary conditions, the modes generating superconformal transformations are given by

$$L_n = \oint e^{in\sigma} T = :\frac{1}{2}\sum_p \alpha^\mu_{n-p}\alpha_{p\mu} + \frac{1}{2}\sum r b^\mu_{n-r} b_{r\mu}: + \underbrace{h_F \delta_{n,0}}_{\text{normal ordering}}$$

$$G_r = \sum_{p\in\mathbb{Z}} \alpha^\mu_p b_{r-p\,\mu}$$

and satisfy

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m}$$

$$[L_n, G_r] = \left(\frac{n}{2}-r\right) G_{n+r} \qquad \left(\text{weight } \frac{3}{2} \text{ for } G\right)$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s} \qquad \left(r=\pm\frac{1}{2} \text{ are global v.f.}\right)$$

where $c = D + \dfrac{D}{2}$ is central charge and

$$h_F = \begin{cases} 0 & NS \\ \dfrac{D}{16} & R \end{cases}$$ is contribution to conformal weight

from fermionic zero modes (the one from bosonic h_B being

$h_B = \frac{1}{2}\alpha_0^2 \sim p^2$ being contained in $\sum_p$ and

the rest of L_0 being called $\underline{N}$ the level as before.

$=$

To understand a rationale, we record that the ground state energy (eigenvalue of $|\Omega\rangle$ under $H = L_0 - \frac{c}{24}$) for a single set of bos + ferm. oscillators ($c = \frac{3}{2}$) is:

$$\underline{NS}: \quad \frac{1}{2}\underbrace{\sum_{\mathbb{Z}} n}_{=-\frac{1}{12}} - \frac{1}{2}\underbrace{\sum_{\mathbb{Z}+\frac{1}{2}} r}_{\frac{1}{24}} = -\frac{1}{24} - \frac{1}{48} = -\frac{3}{2}\frac{1}{24}$$

$$\rightsquigarrow \quad L_0 |\Omega\rangle = 0$$

$$\underline{R} \quad \frac{1}{2}\sum_{\mathbb{Z}} n - \frac{1}{2}\sum_{\mathbb{Z}} r = 0 \quad !$$

$$\rightsquigarrow \quad L_0 = \frac{c}{24} = \frac{1}{16}.$$

⌈ The fastest to remember this is via the existence of a "scalar" supercharge G_0 of $G_0^2 = 2H$ in the "spin 0" Ramond sector, but perhaps this is not useful yet. ⌋

② Constraints

- We intend to define the "$\mathcal{N}=(1,1)$ superstring" by imposing
$$L_n = G_r = 0 \qquad \tilde{L}_n = \tilde{G}_r = 0$$
(of which $G_r = \tilde{G}_r = 0$ will suffice because of $\{G_r, G_s\} = 2L_{r+s}$)

(if we can, depending on D) on a set of sectors (representations of super Virasoro) compatible with defining interactions (in particular, super modular invariance).

- This will shift the ground state discussion (including L_0 constraint and G_0 if present) around depending on the scheme.

light-cone gauge: infinite oscillator contributions only from transverse direction, zero-modes from all for open strings, schematically, $H=0$ constraint leads to mass formula

$$m^2 = -p^2 = \frac{1}{\alpha'}\left(N^\perp - \underset{\overset{\parallel}{\frac{3}{2}\frac{d-1}{24}}}{\frac{d-1}{16}}\right) \qquad N^\perp \in \{0, \tfrac{1}{2}, 1, \dots\}$$

in NS sector

$$= \frac{1}{\alpha'}(N^\perp + 0) \qquad N^\perp \in \{0, 1, 2, \dots\}$$

in R sector

the latter is also square of $G_0 = 0$ constraint.

$$G_0 = \underbrace{\alpha_0^{\mu} \cdot b_{0\mu}}_{\text{all}} + \underset{\sim}{G_0^{\perp}}$$

transverse only ~ m

which remembering $\{b_{0\mu}, b_{0\nu}\} = \eta_{\mu\nu}$ is the advertised "Dirac equation."

<u>old covariant quantization</u> constants adjusted s.t. subquotient $\underbrace{\text{positive definite}}$.

<u>BRST</u>: (bosonic) superghosts make contributions to c & h s.t.

at the end all agree on $\mathcal{H}_{\text{superstring}}$

- level matching $L_0 = \tilde{L}_0$ still operates for the closed string;
- whence the half-integer ~~modding~~ levelling of $N^{\perp}, \tilde{N}^{\perp}$ in NS
- becomes a problem. To alleviate this, the $\mathbb{Z}/2$ grading by worldsheet fermion number plays a central role.

$$(-1)^F b_r = - b_r (-1)^F \qquad (-1)^F \alpha_n = \alpha_n (-1)^F$$

(3) Neveu-Schwarz (and Neveu-Schwarz)

- There are no fermionic zero modes so that Fock space ground state is labelled simply by spacetime momentum $p^\mu \sim \alpha_0^\mu \sim \tilde{\alpha}_0^\mu$, which the L_0 constraint makes tachyonic

$$m^2 = -\frac{1}{\alpha'}\frac{d-1}{16} \quad \text{(open)}$$

$$= -\frac{4}{\alpha'}\frac{d-1}{16} \quad \text{(closed)}$$

- The level $\frac{1}{2}$ states for the open string

$$\varepsilon_\mu \psi^\mu_{-1/2}|p\rangle$$

are required by $G_{1/2} \sim \alpha_0^\mu \psi_{1/2\,\mu} + \cdots$ constraint to be transverse $\varepsilon_\mu p^\mu = 0$, and are massless and positive definite modulo $\varepsilon^\mu \sim p^\mu$ if

$$p^2 = \frac{1}{\alpha'}\left(\frac{1}{2} - \frac{D-2}{16}\right) = 0$$

i.e. $D = 10$. from now on!

- For the closed string, ~~NS~~ NS-NS sector!

$$\psi^\mu_{-1/2}\tilde{\psi}^\nu_{-1/2}|p\rangle$$

give physical states of massless graviton, $B_{\mu\nu}$, dilaton.

- The crucial difference to the bosonic string is that the tachyonic ground state and first excited massless level have <u>opposite worldsheet fermion number</u> $(-1)^F$ (and also $(-1)^{\tilde{F}}$, $(-1)^{F+\tilde{F}}$ being the same b/c level matching).
- This being a symmetry, standard consideration imply that the $(-1)^F = +1$ eigenspace is closed under interactions, and one can project to it like we did in orbifolds.
- For "miraculous" reasons it turns out that in this context, the "correct assignement" with our definition is $(-1)^F |NS \text{ ground state}\rangle = -|p\rangle$.

Then projecting onto states with $(-1)^F = +1$ removes the tachyons from the spectrum so that the lowest lying modes are massless.

$\rightarrow$ famous GSO projection (Gliozzi - Scherk - Olive 1976)

physical states = 8 (open string) (NS+)

$= 64 = 35 + 28 + 1$ (closed string) (NS+, NS+)

$35 = \frac{8 \cdot 9}{2} - 1$ (symmetric traceless)

$28 = \frac{8 \cdot 7}{2}$ (antisymmetric)

1 (trace).

4 Ramond

There are fermionic zero modes $\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$ $\mu,\nu = 0, .. d$

(possibly also $\{\tilde{b}_0^\mu, \tilde{b}_0^\nu\} = \eta^{\mu\nu}$)

Facts: (special to $D=10$ being $\equiv 2 \mod 8$)

- Starting from $\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$

$$\sigma^2 = \sigma^4 = \sigma^6 = \sigma^8 = i\rho^0$$

$$\sigma^3 = \sigma^5 = \sigma^7 = \sigma^9 = \rho^1$$

we define 32×32 -dimensional matrices

$$\gamma^{0,1} = \rho^{0,1} \otimes 1 .. \otimes 1$$

$$\gamma^{2,3} = \rho^0 \rho^1 \otimes \sigma^{2,3} \otimes .. \otimes 1$$

$\vdots$

satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and (because $i^4 = 1$) operating on a real vector space S', which via

$$\Gamma^1 = \gamma^0 + \gamma^1 \quad \bar{\Gamma}^1 = -\gamma^0 + \gamma^1$$

$$\Gamma^2 = \gamma^2 + i\gamma^3 \quad \bar{\Gamma}^2 = \gamma^2 - i\gamma^3$$

$\vdots$

$$\{\Gamma^i, \bar{\Gamma}^j\} = 2\delta^{ij}$$

Can be understood as "realisable" Fock space of 5 complex fermions

$$S' \otimes \mathbb{C} = (\Lambda^{\bullet}\mathbb{C}^5)$$

and thereafter splits (over $\mathbb{R}$) as $S = S^+ \oplus S^-$ according to the parity of number of Γ's / γ's.

- The $S^{\pm}$ are 16-dim. Majorana-Weyl representations of $SO(1,9)$ generated by

$$J^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \qquad \text{(check this!)}$$

and under an identification $b_0^\mu \leftrightarrow \gamma^\mu$ as representation spaces of fermionic zero modes are distinguished by $(-1)^F$. (Clearly, no exact meaning which is which but let's say S^+ has $(-1)^F = +$.)

- The GSO projection for open strings selects S^+ as subrep. so that labelling basis of

$$S^+ \quad u = 1, .. 16$$

$$S^- \quad v = 1, .. 16$$

we find 16 Ramond ground states $|p, u\rangle$ on ~~of~~ which L_0 constraint ~~gives~~ $p^2 = 0$

is now by construction litterally the square of massless Dirac equation from G_0 constraint

$$0 = \sum_u G_{0\,v}^{\;u} |p,u\rangle$$

$$\gamma^\mu = \begin{pmatrix} 0 & \gamma^{\mu\;v}_{\;u} \\ \gamma^{\mu\;u}_{\;v} & 0 \end{pmatrix}$$

$$= \sum_u p_\mu \gamma^{\mu\;u}_{\;v} |p,u\rangle$$

In a rest frame $p_\mu = (p_0, p_1 = p_0, 0, \ldots 0)$ this says

$$\Gamma^{+} = \gamma^0 + \gamma^1 \; |p,u\rangle = 0$$

meaning only $\Gamma^{2,3,4,5}$ are excited.
(even number of)

⇝ Massless Physical states of open string in Ramond sector form 8-dim. repres. of massless little group SO(8). 8_{s^+}

· Together with transverse vector modes from GSO projected NS sector in 8_V, they form physical states of 10d $\mathcal{N}=1$ vectormultiplet.