Lecture 21 Orbifolds

27.1.2022

The orbifold construction plays an important role in relating string backgrounds based on free fields with "less well-understood" models with reduced symmetry. For us, it will also be an occasion to reflect on the importance of modular invariance and illustrate yet another way in which "strings see geometry differently than point particles".

<u>Note:</u> We'll only consider closed strings and omit most calculations. For details, see e.g. Polchinski Chapter 8.5.

① What?

Orbifolds are spaces with certain (fairly mild) types of singularities.

Intuitively:

- smooth manifolds are (topological) spaces locally modelled on euclidean spaces $\mathbb{R}^n$.
- orbifolds are topological spaces locally modelled on quotients of euclidean space by "good" actions of (finite) group.

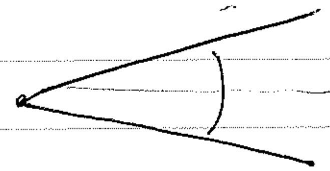
<u>Examples:</u> Global quotients of manifolds by the action of discrete groups with finite isotropy group / s.t. quotient is Hausdorff.

M/Γ

In the physics context (with gravity), Γ should of course act by isometries.

· Concretely, $\mathbb{R}^2/\mathbb{Z}/n$ where $k \in \mathbb{Z}/n$ acts by rotation by $2\pi k/n$ is an orbifold that can be identified with cone of opening angle

"orbifold singularity"



2θ $\qquad$ $\sin\theta = \frac{1}{n}$.

· The arguably simplest compact example is

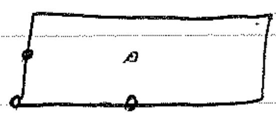
$$S^1_R/\mathbb{Z}/2 \cong I = [0, \pi R] = \mathbb{R}/_{2\pi R \rtimes \mathbb{Z}/2}$$

where non-trivial element $\rho \in \mathbb{Z}/2$ acts by

all we analyze. $\qquad \rho X = -X$.

12 orbifold singularities

· $T^2/\mathbb{Z}/2$ where T^2 is $S^1_{R_1} \times S^1_{R_2}$ rectangular (doesn't have to be) and $\mathbb{Z}/2$ acts by simultaneous reflection on two coordinates is "pillowcase" with 4 singularities $\mathbb{R}^2/\mathbb{Z}/2$ at $(0,0)$, $(\pi R_1, 0)$, $(0, \pi R_2)$, $(\pi R_1, \pi R_2)$ (two-torsion pts of elliptic curve).



② Particles

- Saying that M/Γ is "M with action of Γ" is more/different information than just giving the quotient space (and this is then true a fortiori for orbifolds that are not global quotients).

- For example, by saying "a particle propagates on $S^1_R/\mathbb{Z}_2$" we mean to consider wavefunctions that are invariant under natural action of ρ on Hilbert space of parent theory, which can be done by projection.

$$\mathcal{H}_{S^1_R/\mathbb{Z}_2} = \mathcal{H}^{\rho}_{S^1_R} = \frac{1+\rho}{2}\,\mathcal{H}_{S^1_R}$$

$$= \left\langle\ e^{ipx} + e^{-ipx},\quad p = \frac{n}{R}\ \right\rangle$$

- Namely, wavefunctions satisfy Neumann b.c. whereas from quotient pt of view (q.m.) Dirichlet might do just as well / be preferred.

- Projection is a good idea because it ensures consistency of interactions (if these are invariant in parent theory, we can obtain consistent interactions in quotient. But we won't discuss this here.

Instead, we record that the worldline partition function (one loop integrand)

$$Z_{S^1/\mathbb{Z}_2}(t) = \mathrm{tr}_{\mathcal{H}_{S^1/\mathbb{Z}_2}}\, e^{-tH} \qquad \left(H = -\partial_x^2 + m^2\right)$$

$$= \mathrm{tr}_{\mathcal{H}_{S^1}}\, \frac{1+\rho}{2}\, e^{-tH}$$

can be written in path-integral form as

$$= \frac{1}{2}\left(Z_{S^1} + Z^{(\rho)}_{S^1} \right)$$

where $$Z_{S^1} = \int_{X(\tau+t) = X(\tau+t) \bmod 2\pi R} DX \; e^{S}$$

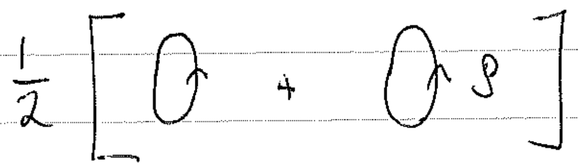
↑ implements projection onto integral momenta

$$Z^{(\rho)}_{S^1} = \int_{X(\tau+t) = -X(\tau+t) \bmod 2\pi R} DX \; e^{S}$$

$$= \mathrm{tr}\, \rho\, e^{-tH}$$

Pictorially

$\simeq$



- In contrast for example cones with irrational opening angle are not orbifolds and consistency not immediately clear. Though for particles intuitively isolated singularities are OK at finite energy, one still requires b.c. in general.
In any case, orbifold singularities are mild much milder than generic (curvature) singularities in GR.

③ Strings

Following the same logic for the string, we (first) project Fock space of theory on S^1_R

$$\mathcal{F}^{25} = \mathcal{F} = \bigoplus_{p=\frac{n}{R}} \mathcal{F}_{p,w} \qquad w = mR/\alpha'$$

$$= \left\langle \prod_{k>0}' \alpha_{-k}^{\lambda_k} \, \tilde{\alpha}_{-k}^{\tilde{\lambda}_k} \, |p,w\rangle \right\rangle$$

onto states that are invariant under action of ρ given by

$$\left.\begin{aligned} \rho\,|p,w\rangle &= |-p,-w\rangle \\ \rho(\alpha_{-k}) &= -\alpha_{-k} \\ \rho(\tilde{\alpha}_{-k}) &= -\tilde{\alpha}_{-k} \end{aligned}\right\} \text{ dictated by mode expansion of } \rho X(\tau,\sigma)\rho^{-1} = -X(\tau,\sigma)$$

which leads to even/ odd combinations depending on parity of $\sum \lambda_k + \tilde{\lambda}_k$. (NB this is not the total level)

$\to$ spectrum can be worked out, mass formula unchanged since invariant under $(p, w) \to (-p, -w)$.)

Second, however, we (need to; see below) include a new, so-called twisted sector, in which X is only periodic up to $\mathbb{Z}/2$ action

$$X(\tau, \sigma + 2\pi) = - X(\tau, \sigma) \mod 2\pi R.$$

The eoms are same as before, but somewhat like DN b.c. for open strings we obtain half-integer mode expansion

$$X = x_0 + i\sqrt{\frac{\alpha'}{2}} \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{\alpha_k}{k} e^{-ik\sigma^-} + \frac{\tilde{\alpha}_k}{k} e^{-ik\sigma^+}$$

with $[\alpha_k, \alpha_l] = k \delta_{k+l}$ etc.

no zero modes $\alpha_0, \tilde{\alpha}_0$, but two possibilities

$x_0 = 0 \mod 2\pi R$, $\pi R \mod 2\pi R$.

(geometrically, states localised at fixed points.)

· The Fock space

$$\mathcal{F}_\rho = \left\langle \prod_{k=\frac{1}{2},\frac{3}{2},\dots} \alpha_{-k}^{\lambda_k} \, \tilde{\alpha}_{-k}^{\tilde{\lambda}_k} \, |0\rangle_{\pi R} \right\rangle$$

with these oscillators comes Virasoro with $c = \tilde{c} = 1$,

$h = \tilde{h} = \frac{1}{16}$

· The physical Hilbert space is obtained by implementing constraints, after projecting $\mathcal{F}_\rho$ onto ρ-invariant states, still acting by

$$\rho(\alpha_k) = -\alpha_k \qquad \rho(\tilde{\alpha}_k) = -\tilde{\alpha}_k$$

$$|0\rangle \mapsto |0\rangle \qquad |\pi R\rangle \mapsto |\pi R\rangle$$

schematically, two "identical copies"

$$\mathcal{H}^{\text{orbifold}} = \mathcal{F}^{\mathbb{Z}/2} + \mathcal{F}_\rho^{\mathbb{Z}/2}$$

→ definitely new states that can't be easily studied go beyond free fields really.

Short of studying interactions, the consistency / necessity of this prescription becomes apparent by consideration of

④ Modular invariance

of (one-loop) partition function. (for orbifold X only)

(orbifold X only).

$$Z = \mathrm{tr}_{\mathcal{F}}\left(\frac{1+\rho}{2}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{c}{24}}\right) + \mathrm{tr}_{\mathcal{F}_\rho}\left(\frac{1+\rho}{2}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{c}{24}}\right)$$

$$= \frac{1}{2}\Big(\underbrace{1\,\underset{1}{\square} + \rho\,\underset{1}{\square}}_{\text{"untwisted sector"}} + \underbrace{1\,\underset{\rho}{\square} + \rho\,\underset{\rho}{\square}}_{\text{twisted sector}} \Big)$$

$q = e^{2\pi i \tau}$

where

$$\beta\,\underset{\alpha}{\square} = \int \mathcal{D}X\, e^{-S(X,\tau)}$$

euclidean path integral on torus mod. par. τ, coordinate z

$$X(z+1) = \alpha X(z)$$
$$X(z+\tau) = \beta X(z)$$

$$\alpha, \beta = 1, \rho \in \mathbb{Z}/2$$

requires twisted sectors: (heuristically)

$$\tau \mapsto \tau + 1: \qquad \beta\,\underset{\alpha}{\square} \longrightarrow \beta\alpha\,\underset{\alpha}{\square}$$

$$\tau \to -\frac{1}{\tau} \qquad \beta\,\underset{\alpha}{\square} \longrightarrow \alpha\,\underset{\beta}{\square}$$

In formulas,

$$ {}_{1}\underset{1}{\square} = \underbrace{\frac{1}{(q\bar q)^{1/24} \left|\prod_n (1-q^n)\right|^2}}_{|\eta(\tau)|^2} \cdot \sum_{n,m} \bar q^{\,p_L^2/2}\, q^{\,p_R^2/2} $$

$$ p_{L,R} = \frac{1}{\sqrt{2}}\left(\frac{n}{R} \pm \frac{mR}{\alpha'}\right) $$

$$ {}_{g}\underset{1}{\square} = $$ (only $p=w=0$ contributes in trace, each oscillator gets -1 ground state energy unchanged at $-\frac{1}{24}$)

$$ = \frac{1}{(q\bar q)^{1/24} \left|\prod_n (1+q^n)\right|^2} $$

$$ {}_{1}\underset{g}{\square} = $$ (half-integer moded oscillators, g.s. energy $-\frac{1}{24}+\frac{1}{16} = \frac{1}{48}$)

$$ = \frac{2\,(q\bar q)^{1/48}}{\left|\prod_{n=1}^{\infty} (1-q^{n-\frac12})\right|^2} $$

$$ {}_{g}\underset{g}{\square} = \frac{2\,(q\bar q)^{1/48}}{\left|\prod (1+q^{n-\frac12})\right|^2} $$

$1 \underset{1}{\square}$ is modular invariant by itself, but $\rho \underset{1}{\square} + 1 \underset{\rho}{\square} + \rho \underset{\rho}{\square}$ only in (this) combination.

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(5) Moduli spaces

- The above considerations generalize to other discrete groups and illustrate general features of "quotienting".

- For example, already winding modes of strings on S^1_R can be viewed as twisted sectors of $\mathbb{R}/_{2\pi R\mathbb{Z}}$ quotient. (and restriction to integer p as invariant states in untwisted sector).

- Even more simply, circle of radius $R/2$ is $\mathbb{Z}/2$ orbifold by $s: X \to X + \pi R$ of circle of radius R.

- In this spirit, starting from self-dual radius $R = 1$ ($\alpha' = 1$) we can consider two a priori distinct $\mathbb{Z}/2$ orbifold.

$$S^1_{R=1} / _s = S^1_{R/\frac{1}{2}} = S^1_{R=2}$$

and

$$S^1_{R=1} / \rho$$

and see that they are in fact equal as quantum theories because of $SU(2)\times SU(2)$ symmetry at self-dual radius.

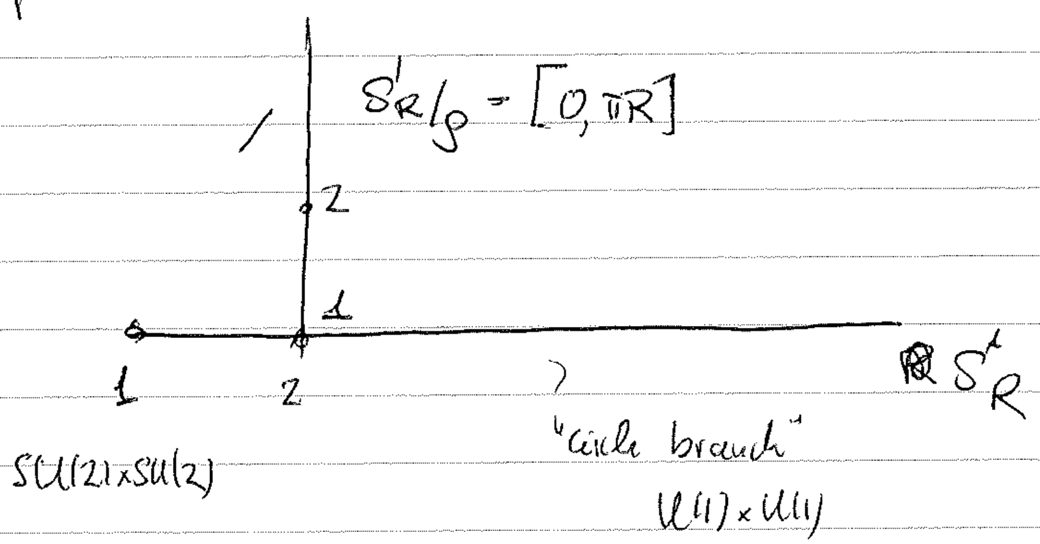
$$s:\ \left[\underset{j_z}{\partial X},\ \underset{j_{x,y}}{e^{2iX_R}\pm e^{-2iX_R}}\right] \longrightarrow \left[\partial X,\ \mp\left(e^{2iX_R}\pm e^{-2iX_R}\right)\right]$$

"rotation by π around z-axis".

$$\rho:\ \left[\partial X,\ e^{2iX_R}\pm e^{-2iX_R}\right] \longrightarrow \left[-\partial X,\ \mp\left(e^{2iX_R}\pm e^{-2iX_R}\right)\right]$$

"rotation by π around y-axis".

This leads to the following (not quite complete) remarkable picture of moduli space of $c=1$ circle + orbifold theories



Lecture 22 Supersymmetry

Historically, supersymmetry was invented on the string worldsheet in order for the spectrum to include spacetime fermions. ~~After~~ ~~a~~ (Ramond, Neveu-Schwarz). After a while, it was realised that by doing things right (namely, ensuring spacetime supersymmetry), one can also solve the remaining "bizzie" that plagued the bosonic string, namely ensure vacuum stability and absence of tachyons.

Today: general ideas, simplest example.
Next week: the real thing.

① Squareroots

To begin with, the string Hilbert space is defined by implementing Virasoro constraints on 2d CFT. Most importantly, mass shell condition

$$(L_0 - 1)\,|\text{physical state}\rangle = 0$$

originates by gauging

② <u>Representations</u> In general, Susy reps are complicated

The relation to stability is best understood ~~b~~ in one dimension, susy QM.

Two words about superstrings

i) (P. Ramond) One may identify the Virasoro constraint

$$(L_0 - 1) \mid \text{physical states} \rangle = 0$$

as a Klein-Gordon equation in spacetime. In particular, L_0 is quadratic in spacetime momenta. On the other hand, the Virasoro constraint is really a consequence of reparametrization invariance of the worldsheet.

$$L_0 \sim p \cdot p = \alpha_0 \cdot \alpha_0$$

P. Ramond realized that if we want Dirac equation for certain states of the string, then this must come from a "squareroot" of reparametrization invariance.

$$G \sim p \cdot \gamma = \alpha_0 \cdot \psi_0$$

$$G^2 \sim L_0$$

Worldsheet supersymmetry is the natural squareroot of reparametrization invariance: As α_n's are modes of bosonic field on the worldsheet, so ψ_n's are modes of some fermionic field ψ ~~The supersymmetric~~

$$\int (\partial X)^2 + \psi \partial \psi$$

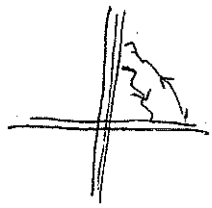
The key is that ~~if~~ since reparametrization invariance is $\delta X = \partial X$, we can arrange a squareroot by postulating

$$\delta X = \psi \qquad \delta\psi = \partial X$$

$$\text{so} \quad \delta^2 X \sim \partial X \qquad \delta^2 \psi = \partial\psi$$

and the action is invariant (up to a total derivative)
→ birth of supersymmetry.

2) A basic property of superstrings at D-brane intersections is that they lead to chiral fermions.



chiral fermion: helicity is correlated with the gauge charge.

<u>Fact</u>
- states of strings ~~starting~~ beginning/ending on a stack of N branes form fundamental/antifundamental representation of $U(N)$ gauge group living on that stack.
- ~~chirality~~ helicity of spacetime reps is related to orientation of string.

This is in recent years (or not so recent) a popular way to construct "standard model" in string theory (so-called intersecting D-brane models).

~~Chapter 11 Superstrings~~

Historically, the main reason for adding supersymmetry to the string was the desire to describe fermions: So far the spacetime excitation spectrum only contained bosons (particles of even spin, force carriers). However, in nature we also observe fermions (particles of half-integer spin, matter fields). It turns out that by introducing worldsheet fermions, and worldsheet superreparametrization invariance, our string spectrum will contain spacetime fermions.
As a byproduct, and if we do things right, the (closed and open) string tachyon drops out, and the critical dimension drops from 26 to 10, and in that dimension the string spectrum displays spacetime supersymmetry.

General ideas of supersymmetry are simple:

A Quantum mechanical system has supersymmetry if the Hamiltonian H can be written as the square of a supercharge Q commuting with H.

$$H = Q^\dagger Q + Q Q^\dagger \quad \leftarrow \text{super.}$$

$$[H, Q] = 0 \qquad \leftarrow \text{symmetry}$$

Relativistically, and geometrically, we know that Hamiltonian is expression of time-translation invariance. To get a supersymmetry, we want to have a geometric notion of a "squareroot" of translation. This can be represented on an enlargement of ordinary Minkowski space known as superspace.

<u>Warmup:</u> Supersymmetric harmonic oscillator.

Suppose we have operators $a, a^\dagger$, $b, b^\dagger$ satisfying algebra:

$$[a, a^\dagger] = 1 = aa^\dagger - a^\dagger a = 1$$

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1$$

anti-commutator

(for example represented on the Hilbert space

$$\mathcal{H} = F \otimes \mathbb{C}^2$$

Fock space $= \langle \overline{(a^\dagger)^n |0\rangle} \rangle$; $a|0\rangle = 0$

$$a \cong a \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b \cong 1 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$b^\dagger \cong 1 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

From SuSy lectures (Winter 2011)

The simplest bosonic degree of freedom is the harmonic oscillator,

$$H_b = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$[a, a^\dagger] = 1$$

$a^\dagger$: creation

a: annihilation operator.

$\frac{1}{2}$: ground state energy (Planck)

The simplest fermionic ~~[illegible]~~ degree of freedom is the two-state system spin up, spin down, with Hamiltonian

$$H_f = B\left(\psi^\dagger\psi - \frac{1}{2}\right)$$

$$\psi^2 = 0 \quad (\psi^\dagger)^2 = 0, \quad \{\psi^\dagger, \psi\} = 1$$

$\psi^\dagger$: creation

ψ: annihilation operator

Note that ground state energy for fermions is negative.

For a simple example of symmetry, we could take two harmonic oscillators, with equal frequency:

$$H_{2b} = \omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \qquad [a_1, a_2] = 0 \text{ etc.}$$

this has symmetry $A = a_1^\dagger a_2 + a_2^\dagger a_1$ that transforms the two bosonic dofs into each other. $[A, H] = 0$

For the simplest example of supersymmetry we take a harmonic oscillator together with a 2 state spin system, with $\omega = B = 1$.

$$H = H_b + H_f = a^\dagger a + \psi^\dagger \psi \qquad ([a,\psi] = [a,\psi^\dagger] = 0)$$

This has symmetry $Q = a\psi^\dagger$

$$[Q, H] = \psi^\dagger a + \underbrace{[a\psi^\dagger, \psi^\dagger\psi]}_{= \{a\psi^\dagger, \psi^\dagger\} - \psi^\dagger\{a\psi^\dagger, \psi\}}$$

$$= \psi^\dagger a - \psi^\dagger a = 0.$$

"super-Jacobi identity".

this is not all that remarkable. The key feature is that the Hamiltonian is the square of the supercharge

$$\{Q, Q^\dagger\} = \psi^\dagger a \psi a^\dagger + \psi a^\dagger \psi^\dagger a$$

$$= \psi^\dagger\psi a^\dagger a + \psi^\dagger\psi + \psi\psi^\dagger a^\dagger a = a^\dagger a + \psi^\dagger\psi = H.$$

The other key of course you noticed (first noticed, ~~apparently~~ allegedly, by Pauli) is that ground state energy cancels between boson and fermion. As one of the consequences, e.g., $H \geq 0$, $H|\psi\rangle = 0$ only if $Q = Q^\dagger|\psi\rangle = 0$. Let's come back to SUSY QM at the end of the lecture.

Another way to state the key feature $Q^2 = H$ is that SUSY is an extension (as a kind of squareroot) of time translation, generated by H. It is natural to ask whether something like this is possible in the relativistic context, namely Quantum Field Theory. The fact that it is possible is essentially a consequence of the observation by Dirac that

$$(\gamma^\mu \partial_\mu)^2 = 2\partial_\mu \partial^\mu \qquad (\text{where } \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

~~etc~~.

So $Q^2 = H$ relating bosons & fermions is not totally unfamiliar.

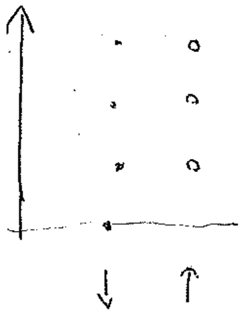
The remarkable thing however, is not just that it is possible, but, it is the only thing you can do to extend ~~[illegible]~~ Poincaré symmetry of relativistic QFT!

- inserted 3 pages from SUSY lectures here.

Spectrum:

$$E_{n\downarrow} = n + \frac{1}{2} - \frac{1}{2} = n$$

$$E_{n\uparrow} = n + \frac{1}{2} + \frac{1}{2} = n+1.$$



All states except for ground state is twice degenerate (Symmetry!).

- There is a Lagrangian formulation of this system:

$$S = \int \left(\frac{\dot{x}^2}{2} \mp \frac{x^2}{2} + i\bar{\psi}\partial_t\psi - \bar{\psi}\psi \right) dt$$

albeit with ψ an anti-commuting variable (Grassmann variable).

$$\Pi_\psi = \bar{\psi} \Rightarrow [\psi, \bar{\psi}] = 1$$

The supersymmetry can be seen as a supervector-field on (x, ψ) space:

$$\delta x \sim \psi$$

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$$\delta x = \bar\psi$$

$$\delta\psi = i\dot{x} + x \qquad \delta\bar\psi = 0$$

$$\delta S = \int \left(\dot{x}\dot{\bar\psi} - x\bar\psi + i\bar\psi\,(i\ddot{x} + \dot{x}) + \bar\psi\,(i\dot{x} + x) \right) dt$$

$$= \int \Big(\underbrace{\dot{x}\dot{\bar\psi} + \bar\psi\ddot{x}}_{=0 \text{ by partial integration}} - x\bar\psi - i\bar\psi\dot{x} + i\bar\psi\dot{x} + i\bar\psi\dot{x} + \bar\psi x \Big) dt$$

$$= 0$$

New Susy:
$$\bar\delta x = \psi \qquad \bar\delta\bar\psi = i\dot{x} - x \qquad \bar\delta\psi = 0$$

$$\delta S = \int \left(\dot{x}\dot{\psi} - x\psi + i\,(i\dot{x} - x)\dot{\psi} - (i\dot{x} - x)\psi \right) dt$$

$$= \int \left(\dot{x}\dot{\psi} - \dot{x}\dot{\psi} - x\psi + x\psi - ix\dot{\psi} - i\dot{x}\psi \right) dt$$

$$\{\delta, \bar\delta\}\, x = \delta\psi + \bar\delta\bar\psi = 2i\dot{x}$$

$$\{\delta, \bar\delta\}\, \psi = \bar\delta(i\dot{x} + x) = i\dot{\psi} + \psi = 2i\dot{\psi}.$$

$$\frac{\delta S}{\delta\bar\psi} = i\partial_t\psi - \psi = 0. \Rightarrow \psi = i\dot{\psi}$$