Lecture 18 Compactification & Duality

11.1.2022

- most glaring deficiencies of our string theory so far:
  - $D \neq 4$
  - no (physical) fermions
  - only abelian gauge symmetry
  - tachyon(s)
- there are (very good!) answers to all of those. Each answer comes with a host of fine print, raises new issues and opens new possibilities
- We begin with extra dimensions.

(1) Kaluza (apparently, following a 1914 suggestion by Nordstrom)

pointed out in 1921 that Maxwell's electromagnetic field A_μ and Einstein's gravitational field $g_{\mu\nu}$, both living on a 4-dimensional spacetime M, can be "unified" by

(i) identifying

$$\begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\mu & 1 \end{pmatrix} =: g^{(5)}$$

as a metric on the 5-dimensional vector space $TM \oplus \mathbb{R}$ at each point in M.

(ii) deriving the Ricci tensor $R^{(5)}$ from $g^{(5)}$ as if spacetime were 5-dimensional $M \times \mathbb{R} = M^{(5)}$ (but neglecting any derivative in the extra direction x^5).

(iii) finding that, schematically,

$$R^{(5)}_{\mu 5} \sim \nabla_\nu F^{\nu}{}_{\mu} \qquad \begin{pmatrix}\text{L.H.S. of} \\ \text{inhomogeneous} \\ \text{Maxwell}\end{pmatrix}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$R^{(5)}_{\mu\nu} = \underbrace{R_{\mu\nu}}_{\text{4-d Ricci}} - \underbrace{T^{(F)}_{\mu\nu}}_{\text{energy-momentum tensor}}$$

while

(iv) 4d gauge invariance $A_\mu \to A_\mu + \partial_\mu \lambda$ is related to 5d reparametrization invariance

$$x^5 \mapsto x^5 + \lambda(x^0, \dots x^3)$$

(though homogeneous Maxwell is not quite Bianchi)

(v) the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0$$

receives an additional contribution proportional to the Lorentz force

$$\Gamma^{\mu}_{5\nu}\, \dot{x}^{\nu}\, \dot{x}^{5} \sim F^{\mu}{}_{\nu}\, \dot{x}^{\nu}\, \frac{p^5}{m}$$

for a (massive) test particle with non-zero Kaluza-Klein momentum $= p^5 =$ electric charge

② Klein (Oscar, not Felix) motivated by the emergence of quantum mechanics, argued in 1926-28 that Kaluza's dynamical assumption $\partial_5 = 0$ should be more satisfyingly replaced by

(i) starting from full-fledged 5-dimensional Einstein theory

$$S^{(5)} = \frac{1}{G_N^{(5)}} \int d^5x \sqrt{-g}\, R^{(5)}$$

possibly coupled to other matter fields

(ii) making the "topological" ansatz

$$M^{(5)} = M \times S^1 \qquad S^1 = \mathbb{R}/_{2\pi} \quad \text{compact manifold.}$$

on which

$$g^{(5)} = \begin{pmatrix} g & 0 \\ 0 & R^2 \end{pmatrix}$$

is a solution of $S^{(5)} \Leftrightarrow R_{\mu\nu} = 0.$

(iii) <u>expanding S^1_R</u> fluctuations in Fourier modes along

· To illustrate the idea, the modes of a (say massless) 5-d field

$$\Phi(x^0,..x^3,x^5) = \sum e^{inx^5} \phi_n(x^0,..,x^3)$$

coupled to $g^{(5)}$ in the standard way, i.e.

$$\delta S^{(5)} \sim \int \sqrt{-g^{(5)}} \left[(\partial_\mu \Phi)^2 + 2 g^{5\mu} \partial_\mu \Phi \partial_5 \Phi + g^{55} (\partial_5 \Phi)^2 \right]$$

behave "from the 4-dimensional point of view" like an infinite tower of scalar fields of

mass $m_n = \left(\frac{n}{R}\right)^2$

and charge $e_n \sim n$ under $A_\mu \sim g_{\mu 5,0}$ as before.

· More generally, starting from a D-dimensional theory (eg. D=26, $g_{\mu\nu}, B_{\mu\nu}, \varphi$ + massive string fields) "making a Kaluza-Klein ansatz" means considering

$$M^{(D)} = \mathbb{R}^{D-k} \times K^{(k)}$$

where $K^{(k)}$ is a k-dimensional <u>compact</u> Riemannian manifold with other background fields satisfying D-dim. eoms and letting "(D-k)-dimensional low-energy physics" be determined by zero-modes of wave operators on K.

3 Some fine (and not so fine) print

- the overall scale of K ($= g_{55} = R^2$ in the example) is in fact among these zero modes, and notoriously annoying. In string theory it mixes with dilaton and the distinction between Einstein and string frame metric, as the last are to fix dynamically, etc.
- for a (somewhat) invariant physical statement, recall that the gravitational coupling constant (D-dim. Newton constant) is dimensionfull

$$[G_N^{(D)}] = L^{D-2}$$

so as to make the action

$$S' = S_{EH}^{(D)} = \frac{1}{G_N^{(D)}} \int \sqrt{-g^{(D)}}\, R^{(D)}\, d^D x$$

dimensionless ($[R^{(D)}] = L^{-2}\ \forall D$), which then under the ansatz

$$g^{(D)} = g^{(D-k)} \oplus g_K$$

becomes

$$S = \underbrace{\frac{1}{G_N^{(D)}} \underbrace{\int_K \sqrt{g_K}\, d^k y}_{Vol(K)}}_{G_N^{(D-k)}} \int R^{(D-k)} \sqrt{-g^{(D-k)}}\, d^{D-k} x$$

Namely $G_N^{(D-k)} = \dfrac{G^{(D)}}{Vol(K)}$ $\leftarrow$ remember

· In the example $K = S^1$, one can also show that the $(D-1)$-dimensional Maxwell coupling is given by

$$g^2_{D-1} = \frac{G_N^{(D)}}{2\pi R} \cdot \frac{1}{R^2} = \frac{G_N^{(D-1)}}{R^2} \quad (\text{dimensionless for } D-1=4)$$

and therefore of gravitational strength, which does not mesh well with natural facts, but also ~~..~~ is not a very general statement as $U(1)$ gauge symmetry depends on isometries of S^1.

· In fact, one of the simplest ideas for non-abelian gauge symmetry in Kaluza-Klein (W. Pauli) is to consider compact mfo. with non-commutative isometry gps. Such mfo. however are not Ricci-flat which is a dynamical problem and also lead to issue of consistency of truncation (solution of low-dimensional ~~..~~ theory does not necessarily lift).

· In string theory

$$G_N^{(D)} = \alpha'^{\left(\frac{D-2}{2}\right)} g_s^2 \qquad \alpha' \sim l_s^2 .$$

· <u>4-d</u>: $\quad G_N^{(4)} = l_p^2 \qquad l_p = 10^{-33}\,\mathrm{cm}$

· K-K-scale

4) Strings on S^1

· In general, string backgrounds of the form $\mathbb{R}^{D-k} \times K^{(k)}$ are defined (in perturbation theory) by quantizing the σ-model action

$$\frac{1}{4\pi\alpha'}\int d^2\sigma \Big(\underbrace{\partial_\alpha X\cdot\partial^\alpha X}_{D-k} + g^{(k)}_{ij}\, \partial_\alpha X^i \partial^\alpha X^j \Big)$$

and interpreted by decomposing the resulting Hilbert space in ~~terms of~~ irreps of $(D-k)$-dimensional Poincaré group.

· As alluded to before, conformal invariance will require that $g^{(k)}$ be Ricci flat, but at least for $R_K \gg l_s$, the high (~~ens~~ worldsheet) energy spectrum will look like D sets of oscillators, so that criticality is still the condition $D=26$.

· For $K = S^1_R = \mathbb{R}/_{2\pi R} \ni X^{25} = X$ we can be completely explicit. The equations of motion are unmodified

$$\partial_+\partial_- X = 0$$

so that all effects are in the boundary conditions on the zero modes.

Classically, the string can wind around S^1.

$$X(\tau, \sigma+2\pi) = X(\tau,\sigma) + 2\pi R\cdot m \qquad m\in\mathbb{Z}$$

while quantum mechanically, we have to require that wavefunctions be single valued on S^1_R:

$$e^{ip(X+2\pi R)} = e^{ipX} \implies p = \frac{n}{R} \qquad n\in\mathbb{Z}$$

In terms of the mode expansion $X = X_L(\sigma^+) + X_R(\sigma^-)$

$$X_L = \frac{1}{2} x_0 + \sqrt{\frac{\alpha'}{2}}\, \tilde{\alpha}_0 (\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum \frac{\tilde{\alpha}_n}{n} e^{-in\sigma^+}$$

$$X_R = \frac{1}{2} x_0 + \sqrt{\frac{\alpha'}{2}}\, \alpha_0 (\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum \frac{\alpha_n}{n} e^{-in\sigma^-}$$

this means

$$\tilde{\alpha}_0 - \alpha_0 = \sqrt{\frac{2}{\alpha'}}\, mR = \sqrt{2\alpha'} \cdot w$$

where $w = \frac{mR}{\alpha'}$ is the "topological charge", winding momentum

while the Noether charge associated with translational invariance is still

$$p = \frac{1}{2\pi\alpha'} \int_0^{2\pi} \dot{X} = \frac{1}{\sqrt{2\alpha'}} \left(\tilde{\alpha}_0 + \alpha_0\right)$$

The Hilbert space decomposes

$$\mathcal{H} = \bigoplus_{n,m} \mathcal{H}_{n,m}$$

where each $\mathcal{H}_{n,m}$ is Fock space for oscillators $\alpha_n, \tilde{\alpha}_n, n \neq 0$ with

$$\tilde{\alpha}_0 \big|_{\mathcal{H}_{n,m}} = \sqrt{\frac{\alpha'}{2}} (p+w) = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} + \frac{mR}{\alpha'}\right)$$

$$\alpha_0 \big|_{\mathcal{H}_{n,m}} = \sqrt{\frac{\alpha'}{2}} (p-w).$$

• The Virasoro algebra satisfied by

$$L_n = \frac{1}{2} \sum :\alpha_p \alpha_{n-p}: \qquad \tilde{L}_n$$

is unmodified, except that after putting back remaining flat uncompactified directions and resolving the constraints e.g. by going to light-cone gauge, the (25-dim) mass formula becomes

$$m_{25}^2 = \frac{4}{\alpha'}\left(\frac{1}{2}\alpha_0^2 + N^{\perp} - 1 \right) \qquad 24 \overset{=23+1}{} \text{ transverse oscillators}$$

$$= \frac{4}{\alpha'}\left(\frac{1}{2}\tilde{\alpha}_0^2 + \tilde{N}^{\perp} - 1 \right)$$

• Level matching: $L_0 = \tilde{L}_0$: $\qquad \alpha_0 = \sqrt{\frac{\alpha'}{2}}\,(p - w)$

$$= \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} - \frac{mR}{\alpha'} \right)$$

becomes "more flexible"

$$\alpha' p \cdot w = n \cdot m = N^{\perp} - \tilde{N}^{\perp}$$

while

$$m_{25}^2 = p^2 + w^2 + \frac{2}{\alpha'}\left(N^{\perp} + \tilde{N}^{\perp} - 2 \right)$$

shows the interpretation of p^2 as KK energy while w^2 is new winding energy, while writing

$$m_{25}^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2 + \text{osc.}$$

is the most succinct manifestation of

5 T-duality

- (The spectrum of) bosonic string compactified on a circle of radius R is isomorphic to that on a circle of radius $R' = \frac{\alpha'}{R}$.

- This extends to a full equivalence of perturbative string theory:

- At the level of worldsheet currents and fields, duality involves the identification

$$\left.\begin{aligned} \partial_+ X' &= \partial_+ X \\ \partial_- X' &= -\partial_- X \end{aligned}\right\} \quad dX' = \overset{\text{2d Hodge star}}{\star}\, dX$$

which can be recognized as a 2d version of electric-magnetic duality in 4d.

$$dA' = F' = \star F = dA$$

under which the Maxwell eqs. w/o source $dF = d\star F = 0$ are invariant.

- At the spacetime level, it is important to realize that T-duality also acts on the string coupling most easily seen from invariance of 25-d. Newton const.

$$G_N^{(25)} = \frac{G_N^{(26)}}{2\pi R} = \frac{g_s^2 \alpha'^{12}}{2\pi R} = \frac{g_s'^2 \alpha'^{12}}{2\pi R'} = G_N'^{(25)}$$

$$\Rightarrow \quad g_s'^2 = g_s^2 \frac{R'}{R} = g_s^2 \frac{\alpha'}{R^2}. \qquad \left(g_s = e^{\Phi_0} \right)$$

The physical interpretation of the duality, and in particular the sense in which the

self-dual radius, $R_X^2 = \alpha'$ $\quad R_X = l_s$

is the "smallest possible compactification radius", can be understood by looking at the

⑥ Low-lying spectrum

· At $n = m = 0$ winding and momentum in X^{25}-direction, we have in addition to

tachyon $\quad |p,0,0\rangle \quad m_{25}^2 = -\frac{4}{\alpha'}$

$g^{(25)}_{\mu\nu}, B_{\mu\nu}, \phi \quad \alpha^i_{-1}\tilde{\alpha}^j_{-1}|p,0,0\rangle \quad m_{25}^2 = 0 \quad i,j = 2,\dots 24$

$\mu,\nu = 0,\dots 24$

two more massless vectors

$$\alpha^{25}_{-1}\tilde{\alpha}^j_{-1}|p,0,0\rangle \qquad i,j = 2,\dots 24$$

$$\alpha^i_{-1}\tilde{\alpha}^{25}_{-1}|p,0,0\rangle$$

which can be interpreted as KK zero modes of $g^{(26)}_{\mu 25} = A^{(1)}_\mu$ and $B^{26}_{25\mu} = A^{(2)}_\mu$ (up to linear combination) and a scalar

$$\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}|p,0,0\rangle.$$

(radius^2 = g_{55} of circle).

· States with $m=0$ zero winding $n \in \mathbb{Z}$ correspond to KK modes of thex. E.g. for the g.s.

$$|p,n,0\rangle \qquad m_{25}^2 = -\frac{4}{\alpha'} + \frac{n^2}{R^2}$$

are tachyonic for large R, small n, but become massive at large n.

· On the other hand, states with $n=0$, $m \in \mathbb{Z}$

$$|p,0,m\rangle \qquad m_{25}^2 = \frac{m^2 R^2}{\alpha'^2} - \frac{4}{\alpha'}$$

are massive for large R, any m, but look like KK tower of T/circle of radius $R' = \frac{\alpha'}{R}$ when R becomes small.

· Namely, it is not so much that compactifications on circles of $R < l_s$ do not exist, but rather that because of our field theory / geometric bias, we will interpret them as compactifications on circles of large radius (b/c they will look like this in experiment).

· Finally, it is interesting to look at states with ~~$n=m=\pm 1$~~ $n \cdot m = \pm 1 = N^\perp - \tilde{N}^\perp$

$$\alpha_{-1}^i |p, \pm 1, \pm 1\rangle \qquad \tilde{\alpha}_{-1}^i |p, \pm 1, \mp 1\rangle$$

$$\alpha_{-1} |p, \pm 1, \pm 1\rangle \qquad \alpha_{-1}^{\theta} |p, \pm 1, \mp 1\rangle$$

which because of

$$m_{25}^2 = \frac{1}{R^2} + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'} = \left(\frac{1}{R} - \frac{R}{\alpha'}\right)^2$$

become massless at the self-dual radius $R^2 = \alpha'$.

and which because of KK-momentum / winding are charged under $U(1)$ gauge fields $A^{(1)}_\mu$, $A^{(2)}_\mu$ and combine with these to form two $SU(2)$ non-abelian gauge fields (or with scalar to form a massive charged vector).

An easy way to see $SU(2)$ structure is to compare OPE of vertex operators

$$J_0 = \partial X^{25} \qquad J_\pm = e^{\pm i p_L X_L}$$

of with $\quad p_L = p + w = \frac{1}{R} + \frac{R}{\alpha'} = \frac{2}{\sqrt{\alpha'}}$

$$h_L = \frac{\alpha'}{4} p_L^2 = 1.$$

with considerations of problem 1 on sheet 7.

Lecture 19 Lattices and Tori

· The topological manifolds $T^k = \underbrace{S^1 \times \ldots \times S^1}_{k \text{ times}} = \left(\mathbb{R}/_{2\pi\mathbb{Z}}\right)^k$

are the simplest higher-dimensional spaces that admit (Ricci-) flat Riemannian metrics and therefore natural compactification manifolds for string theory. Phenomenologically not viable, but still a rich source of theoretical exercises.

1) Metrics

· Viewing

$$T^k = \mathbb{R}^k_y / 2\pi\mathbb{Z}^k \qquad \alpha = 1, \ldots k$$

$$(y^\alpha) \qquad (m^\alpha)$$

we can induce any flat Riemannian metric by pulling back the euclidean inner product on $\mathbb{R}^k_x \ni (x^i)_{i=1,\ldots k}$ by a linear map invertible

$$R = \left(R^i{}_\alpha\right).$$

· In ~~a~~ y-coordinates, the metric is

$$g_{\alpha\beta} = R^i{}_\alpha R^j{}_\beta \delta_{ij} \qquad g = R^T R$$

· In x-coordinates, the metric is diagonal, but ~~ϕ~~ in the quotient

$$T^k = \mathbb{R}^k_X / _{2\pi}\Lambda$$

is by the "non-trivial" full lattice

$$\Lambda = R \cdot \mathbb{Z}^k \ni \left(w^i = R^i_{\alpha} m^{\alpha}\right) \quad i=1\ldots k$$

- In the string context m^α are the winding numbers and we'll call w^i winding momenta setting $\alpha'=1$ to hopefully simplify formulas and avoid confusion with other occurances of α.

- A point-particle's quantum-mechanical wavefunctions

$$\psi(x) = e^{i p_i x^i} = e^{i n_\alpha y^\alpha}$$

are single valued iff momentum numbers $n_\alpha \in \mathbb{Z}$ or equivalently momenta

$$p_i = R_i^{\alpha} n_\alpha \in \Lambda^\vee \qquad \left(R_i^\alpha\right) = R^{-T}$$

take values in the lattice dual to Λ wrt. euclidean inner product, have energy (worldline)

$$\frac{p^2}{2} = \frac{g^{\alpha\beta} n_\alpha n_\beta}{2}$$

and are orthonormal wrt

$$\int_{T^k} \frac{d^k x}{\mathrm{vol}\, T^k)}$$

$$\sim (2\pi)^k \underbrace{|\det R|}_{|\Delta|'}$$

. In KK compactification on T^k, we obtain

k massless vectors $A_\mu \sim h_{\mu\alpha}$

$\frac{k(k+1)}{2}$ massless scalars $\phi_{\alpha\beta} \sim h_{\alpha\beta}$

from fluctuations ~~of~~ the metric. This is already interesting because:

. the charges are not quantized in the same basis in which the metric on the gauge fields is diagonal.

. That metric, as well as the kinetic term for the scalars

$$\sim \quad tr\left(g^{-1}\partial_\mu \phi\right)^2 \qquad \left(\text{cmp } \overset{WP}{\text{metric on}}\ \text{Met}(\Sigma)\right)$$

(in fact depends on these scalars (moduli).
obtained by expansion of the action / Einstein eqs..

The main new physics feature in this lecture however arises from the possibility of coupling the string to a flat Kalb-Ramond B-field background.

② B-field

$$\frac{1}{2\pi}\int_{\Sigma} X^{*}(B) = \frac{1}{2\pi}\int B_{ij}\,\dot X^{i} X'^{j}\, d\tau\, d\sigma$$

$$B_{ij} = -B_{ji} \qquad b_{\alpha\beta} = R^{i}_{\alpha} B_{ij} R^{j}_{\beta} = -b_{\beta\alpha}$$

- As a total derivative (locally), does not change eoms (check this). However, in analogy with coupling particle to (flat) magnetic potential, it enters the relation between kinematical $\dot X$ and canonical momentum

$$\Pi = \frac{\partial \mathcal{L}}{\partial \partial_\tau X} = \frac{1}{2\pi}\left(\dot X + BX'\right)$$

whose zero mode

$$p = \frac{1}{2\pi}\int_0^{2\pi} d\sigma \left(\dot X + BX'\right)$$

is Noether charge for X-translations and canonical conjugate.

In terms of standard expansion modes left/right = anti/holomorphic

$$X_{L,R} = \ldots \sqrt{\frac{\alpha'}{2}}\,\tilde\alpha_0\, \sigma^{\pm} \ldots$$

$$\tilde\alpha_0 = \sqrt{\frac{\alpha'}{2}}\, p_L$$

we have

$$\alpha_0 = \sqrt{\frac{\alpha'}{2}}\, p_R$$

$$p = \frac{1}{2}\, p_L + p_R$$

$$\sigma^{\pm} = \tau \pm \sigma$$

In terms of standard left/right = anti/holomorphic modes

$$X_{L,R} = \dots \sqrt{\frac{\alpha'}{2}}\, \tilde{\alpha}_0 \sigma^{\pm} \qquad \sigma^{\pm} = \tau \pm \sigma$$

$$= \dots \sqrt{\frac{\alpha'}{2}}\, p_{L,R}\, \sigma^{\pm} \dots$$

we have

$$p = \frac{1}{\sqrt{2}}\left(p_L + p_R + B\,(p_L - p_R)\right)$$

while $w = \frac{1}{2\pi\alpha'} \int_0^{2\pi} X' \, d\sigma = \frac{1}{\sqrt{2}}(p_L - p_R)$

$$p_L = \frac{1}{\sqrt{2}}\left(p - Bw + w\right)$$

$$p_R = \frac{1}{\sqrt{2}}\left(p - Bw - w\right)$$

Contribution to g.s. energy via $L_0 = \frac{\alpha_0^2}{2}$

is $h_{L,R} = \frac{1}{2} p_{L,R}^2$

$$h_L + h_R = \frac{1}{2}\left((p - Bw)^2 + w^2\right)$$

$$h_L - h_R = (p - Bw)\cdot w = p\cdot w \qquad \text{b/c } B \text{ is a.s.}$$

$\in \mathbb{Z}$ iff $w \in \Lambda = R\,\mathbb{Z}^k$

$p \in \Lambda^\vee = R^{-1}\mathbb{Z}^k$

$\rightsquigarrow$ Level matching possible. Analysis of spectrum possible and interesting.

3 Charges

· Always present in addition to k Kaluza-Klein $U(1)$ gauge bosons $A^{(1)}_\mu \sim h_{\mu\alpha}$ are k Kalb-Ramond vector bosons $A^{(2)}_\mu \sim B_{\mu\alpha}$. The number of massless scalars goes up to

$$\frac{k(k+1)}{2} + \frac{k(k-1)}{2} = k^2$$

and there are special loci with various non-abelian gauge symmetry in the space they span, and additional discrete identifications.

· The analysis is expedited by studying instead of the pair $(\Lambda \sim R\mathbb{Z}^k, B)$ the full set of charges

$$\Gamma = \left\{ \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} R^{-T} & R-BR \\ R^{-T} & -R-BR \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \right\}$$

$$= S \cdot \mathbb{Z}^{2k}$$

as a full lattice in $\mathbb{R}^{2k} = \mathbb{R}^k \oplus \mathbb{R}^k$, equipped with inner product

$$(p_L, p_R) \circ (p_L', p_R') = p_L p_L' - p_R p_R' = pw' + p'w = nm' + n'm$$

of signature (k, k).

Fact: W.r.t. this inner product, Γ is even ($p \cdot p \in 2\mathbb{Z}$) and self-dual $\Gamma^\vee = \Gamma$ and all even and self-dual lattices up to isomorphism in $\mathbb{R}^{k,k}$ arise in this way.

④ Moduli space

- Changing basis $(n,m) \to \left(\frac{n+m}{\sqrt{2}}, \frac{n-m}{\sqrt{2}}\right)$ ("standard lattice"), S' is an isometry of $\mathbb{R}^{k,k}$,

$$S' \in O(k,k;\mathbb{R}).$$

- Rotating $\mathbb{R}^k_L$ or $\mathbb{R}^k_R$ separately ~~do~~ does not change string spectrum and lattices are (considered) isomorphic.
- discrete identification act on standard lattice by

$$(n,m) \to (n+Nm, m) \quad \text{where } N^T = -N$$

is antisymmetric integral

$$(n,m) \to (m,n)$$

generating $O(k,k;\mathbb{Z})$.

Upshot: The (Narain) moduli space of toroidal compactifications of bosonic string is the homogeneous space.

$$\boxed{\frac{O(k,k;\mathbb{R})}{O(k)\times O(k,\mathbb{R})} \Big/ O(k,k;\mathbb{Z})} \quad \mathcal{M}_k$$

including the natural homogeneous metric (of typically negative curvature).

Example: , $k=2$.

$$\mathcal{M}_k = \mathbb{H} \times \mathbb{H} \, / \, \mathbb{Z}_2$$

first factor : complex structure $\tau = \tau_1 + i\tau_2$ (cmp. worldsheet)
second factor : complexified Kähler structure
$$\rho = \rho_1 + i\rho_2 = B + iV.$$

with exchange symmetry $\tau \longleftrightarrow \rho$ (mirror symmetry)
and metric
$$\frac{d\tau^2}{\tau_2^2} + \frac{d\rho^2}{\rho_2^2}$$

The lattice description is also useful for understanding

5 Modular invariance

of one-loop integrand. Torus partition function

$$Z = \mathrm{tr}\; q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} \qquad q = \exp(2\pi i \tau)$$

τ: worldsheet complex structure

$\mathrm{Im}\, \tau_2 > 0.$

$$= Z_{\text{noncompact}} \cdot Z_{T^k}$$

$$Z_{T^k} = |\eta|^{-2k} \sum_{(p_L, p_R) \in \Gamma} e^{2\pi i \left(\tau \frac{p_R^2}{2} - \bar{\tau} \frac{p_L^2}{2}\right)}$$

convergent

$$\eta(q) = q^{1/24} \prod (1-q^n)$$

$$\eta\left(-\tfrac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau)$$

via Poisson summation

$$\sum_{\gamma \in \Gamma} f(\gamma) = \frac{1}{|\Delta|} \sum_{\gamma^\vee \in \Gamma^\vee} \hat{f}(\hat{\gamma})$$

$= 1$ by self-duality

$$f \sim e^{2\pi i \tau p^2} \qquad \hat{f} \sim \frac{1}{\sqrt{\tau}} e^{2\pi i\, p^2/2}$$

$$\Rightarrow \quad Z\left(-\frac{1}{\tau}\right) = Z(\tau)$$

$Z(\tau+1) = Z(\tau)$ by evenness of lattice.

$$p_R^2 - p_L^2 \in 2\mathbb{Z}.$$