Lecture 17 (One) Loop amplitudes

- Last time, we ended with the following "beautiful" formula for perturbative (closed) string amplitudes

$$A \sim \sum_g g_s^{2g-2+n} \int_{\mathcal{M}_{g,n}} \int e^{-S_P - S_{gh}} \prod (b, \mu_k) \prod_{\text{fixed}} cV_j \prod_{\text{unfixed}} V_i \, \mathcal{D}X \mathcal{D}b \mathcal{D}c \; d\sigma^i dt^k.$$

- For open/unoriented strings, we have to sum also over the number of boundaries / crosscaps an in general there are several worldsheet topologies contributing at each order in g_s-expansion.

- Main physics features so far: · no extra data needed no free coupling constant besides g_s (Operator-state correspondence).
  - tree amplitudes exhibit "Dolan-Horn-Schmid" duality and reproduce field theory results expected from towers of Regge trajectories (graviton, photon, etc.), but do not agree with parton structure of QCD at high energies.

- Needless to say the general evaluation of loop amplitudes is complicated, even for free (quadratic) worldsheet theories, as it requires:

- coordinates and local measure on Teichmüller space $\mathcal{T} = \mathrm{Met}/_{\mathrm{Diff}_0 \ltimes \mathrm{Weyl}}$
- understanding of Mapping class group and singularities of $\mathcal{M} = \mathcal{T}/\mathrm{MCG}$
- Scalar expectation values (Green's functions) and determinants on compact genus g Riemann surfaces.

- All of these are within reach of complex/harmonic analysis and in principle also of numerical study. For now, share two folklore facts

  - At each loop order, the amplitudes are "finite". More precisely, moduli spaces are compact. (finite volume) Singularities of integrand at boundaries (degeneration of Riemann surfaces) can be interpreted as IR divergences in spacetime (non-existence of S-matrix due to long-range interactions from exchange / emission of massless particles)
    WRONG: Bosonic string also has genuine divergences signalling tachyonic instability; goes away in superstring.

  - genus-g contributions grow with g asymptotically like $\mathrm{vol}_{WP}(\mathcal{M}_g) \sim (2g)!$, signalling non-perturbative effects / ambiguities of order $\exp\left(\sqrt{\frac{1}{g_s^2}} = \frac{1}{g_s}\right)$ (D-branes).

In field theory (Yang-Mills) amplitudes typically grow as $l!$ with loop order, non-perturbative effects of order $\exp\left(\frac{1}{g_{YM}^2}\right)$. (instantons).

(non-perturbative effects are stronger in string theory)

Goal today: Fix a few ideas at one loop, but even here can't be fully precise because we ignored some important normalization issues. We begin with another reference to QFT that will allow interpretation of results.

① Vacuum amplitudes

- In QFT, the "vacuum amplitude" or "partition function"

$$Z = \int \mathcal{D}\Phi \, e^{-S[\Phi]}$$

disappears as normalization factor from (perturbative) correlation functions defined by formulas such as

$$\langle \underbrace{\phi \ldots \phi}_{n} \rangle = \frac{\partial^n}{\partial J^n} \ln Z[J] \Big|_{J=0} \qquad Z[J] = \int \mathcal{D}\Phi \, e^{-S + \Phi_i J}$$

(But for that same reason Z is in some sense all that is needed.)

- With compact euclidean time, it can be interpreted as thermal partition function

$$Z_\beta[\Phi] = \int_{\Phi(x^0+\beta) = \Phi(x^0)} \mathcal{D}\Phi \, e^{-S[\Phi]} = \operatorname{tr}_{\mathcal{H}_T} e^{-\beta H}$$

which alone justifies its physical interest.

- Z is also of physical interest for coupling to gravity, as was illustrated by the 2-d Weyl anomaly on the worldsheet.
- To see what it teaches about space-time physics, we observe that for a free scalar field in D dimensions

$$Z = \int \mathcal{D}\Phi \, \exp\left(-\frac{1}{2}\int d^D x \, \left(\partial_\mu \Phi \, \partial^\mu \Phi + m^2 \Phi^2 \right)\right)$$

$$\sim \left(\det\left(-\partial^2 + m^2\right)\right)^{-1/2} = \exp\left(-\frac{1}{2} \operatorname{tr} \ln\left(-\partial^2 + m^2\right)\right)$$

$$= \exp\left(-\frac{1}{2} \underbrace{\int d^D p \, \ln\left(p^2 + m^2\right)}_{\text{UV divergent}}\right)$$

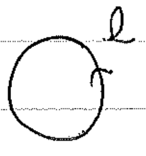
is given, in the worldline formalism, entirely by multi-particle one-loop contributions

$$Z = \sum \frac{Z_1^n}{n!}$$

where (ignoring factors of 2 and 2π)

$$Z_1 = \int d^D p \, \ln\left(p^2 + m^2\right) = \int d^D p \int \frac{d\ell}{\ell} \, e^{-\ell\left(p^2 + m^2\right)}$$

$$\left(\int d\ell \, e^{-\ell x} = \frac{1}{x} \right)$$



$$= \int \frac{d\ell}{\ell} \operatorname{tr} e^{-\ell\left(-\partial^2 + m^2\right)}$$

is the integral ~~of the~~ over invariant lengths of the trace of the worldline Hamiltonian

$$H = -\partial^2 + m^2$$

We could derive the measure from gauge fixing "particle Polyakov action" and be more careful with other things as well; but should in any case note that trace is over worldline Hilbert space <u>without constraint</u>. What's important is the (physical) intuition that UV divergences of Z (at large p) correspond to short-distance (small l) divergence of Z_1. Specifically

$$Z_1 \sim \int_0^\infty \frac{dl}{l} \frac{1}{l^{D/2}} e^{-lm^2}$$

and that there are (potential) IR divergences at $l \to \infty$ when $m^2 = 0$ (or < 0 :-)).

② <u>In string theory</u>

- In principle, Z might receive contributions from any number of loops, even for a free worldsheet action.
- More precisely, this depends on the proper treatment of the conformal/standard Killing group at $g = 0, 1$.
  - Naively, $\mathrm{Ker}\, P \neq 0$ implies $\det X = 0$ and one should conclude that $A = 0$ with insufficient number of insertions ($2g - 2 + n < 0$). Really however, our original (f.d.) formula never was valid when Φ was not diffeo (ineffective gp action)
  - This could be repaired by dividing CKG from gauge gp beforehand, which works for vacuum amplitudes at $g = 0$ but runs into difficulties of coherence ~~ones~~ with insertions. (Subgps. of CKG fixing points are not normal?)

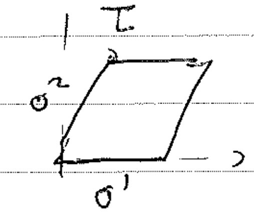
- Most physically, one can think of gauging being associated with "dividing by volume of G", counting only physically inequivalent configurations. Then a zero result is due to a non-compact CKG, while for compact CKG, a non-zero result might obtain even with only an insufficient number of insertions. (det calculus shows that this interpretation is necessary from point of view of Weyl invariance).

This applies in particular to one-loop contribution for closed strings:

$$\mathcal{T}_1 = \mathrm{Met}\left(T^2 = S^1_{\sigma^1} \times S^1_{\sigma^2} = [0,2\pi]^2 \right) / _{\mathrm{Diff}_0 \times \mathrm{Weyl}} = \mathbb{H}$$

where τ, $\mathrm{Im}(\tau) > 0$ corresponds to complex structure induced from identification

$$\Sigma = \mathbb{C} / \mathbb{Z} \oplus \tau \mathbb{Z}$$



and representative flat metric

$$ds^2 = |dz|^2 = |d\sigma^1 + \tau d\sigma^2|^2 = h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta$$

$$\hat{h} = \begin{pmatrix} 1 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & |\tau|^2 \end{pmatrix}$$

of volume $(2\pi)^2 \cdot \mathrm{Im}\,\tau$.

perhaps " τ_2

In this case $CKG = T^2$ is indeed compact, of (non Weyl-invariant) volume

$$\int_{CKG} \det_{\alpha\beta} \int_\Sigma \sqrt{\det \hat{h}}\, \hat{h}_{\alpha\beta} \sim \tau_2^{\,2}$$

(3) Matter x Ghosts

Emulating the point particle, the X path-integral

$$\int \mathcal{D}X\, e^{-S_p(X,\hat{h})} \qquad X(z+1) = X(z+\tau) = X(z) \text{ b.c.}$$

calculates the grand worldsheet partition function before constraints

$$\mathrm{tr}_{\mathcal{H}}\, e^{-2\pi\tau_2 H + 2\pi i \tau_1 P}$$

corresponding to euclidean time evolution by $2\pi\tau_2$ on the cylinder with the worldsheet Hamiltonian

$$\partial_\tau = H = L_0 + \tilde{L}_0 - \frac{D}{12}$$

conformal anomaly $= \frac{c}{24} + \frac{\tilde{c}}{24}$.

and insertion of worldsheet momentum

$$\partial_\sigma = P = L_0 - \tilde{L}_0 \qquad \text{(conformal spin)}$$

because of twist on boundary conditions.

Using that $\mathcal{H} = \int d^D p \ \otimes \mathcal{F}^{\mu}_{p^{\mu}}$ (before constraints) factorises into left/right-moving oscillators (except for zero modes) with

$$L_0 = \frac{\alpha'}{4} p^2 + N \qquad \tilde{L}_0 = \frac{\alpha'}{4} p^2 + \tilde{N}$$

~~and this is~~ and the partition function of a single harmonic oscillator

$$\mathrm{tr}\, e^{-\beta H_{osc.}} = \sum_n e^{-\beta\omega n} = \frac{1}{1-e^{-\beta\omega}}$$

this evaluates to (ignoring a big factor of spacetime volume)

$$\int d^D p \ \mathrm{tr}_{\otimes \mathcal{F}^{\mu}_{p^{\mu}}} \ q^{L_0 - \frac{D}{24}} \ \bar{q}^{\tilde{L}_0 - \frac{D}{24}}$$

where $q = e^{2\pi i \tau}$ $\quad \tau = \tau_1 + i\tau_2$

$$= \int d^D p \ e^{-\frac{\alpha'}{2}\tau_2 p^2} \left(\frac{q^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1-q^n)} \ \frac{\bar{q}^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1-\bar{q}^n)} \right)^D$$

$$\sim \tau_2^{-D/2} \ |\eta(q)|^{-2D}$$

in terms of the Dedekind eta function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n).$$

The ghost/antighost contribution is, similarly,

$$\int e^{-S_{gh}} \mathcal{D}b\,\mathcal{D}c = \mathcal{D}\ \mathrm{tr}_\Lambda\left((-1)^F q^{L_0 - \frac{c^{gh}}{24}} \bar{q}^{\tilde{L}_0 - \frac{c^{gh}}{24}} \right)$$

$$= \Big(\text{two sets of left/right moving fermionic oscillators with periodic b.c. in both space and time on } L_0 |\downarrow\downarrow\rangle = -1,\quad c^{gh} = -26 \Big)$$

$$= \mathcal{D}\ |\eta(q)|^{2}$$

where we cannot ignore the "ghost target volume" $\mathcal{D} \sim \tau_2^2$ because of its τ_2 dependence.

④ Integration

The moduli metric

$$\int_\Sigma \sqrt{h}\ \mathrm{tr}_{\alpha\beta}\left(\partial_\tau h\, h^{-1}\, \partial_{\bar\tau} h\, h^{-1} \right)$$

evaluates for $h = \hat{h} = \begin{pmatrix} 1 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & |\tau|^2 \end{pmatrix}$ to $\dfrac{d^2\tau}{\tau_2}$

so that overall we obtain (with only "mild" cheats, but up to constant factors)

$$Z_1 \approx \int \frac{d^2\tau}{\tau_2} \frac{1}{\tau_2^2} \frac{1}{\tau_2^{D/2} |\eta(q)|^{2D}} \tau_2^2 \, |\eta(q)|^2$$

$= ($ for $D=26)$

$$= \int \frac{d^2\tau}{\tau_2^2} \left(\frac{1}{\tau_2^{1/2} |\eta(q)|^2} \right)^{24} \qquad q = e^{2\pi i \tau}$$

Now crucially, the integrand is invariant under the genus 1 mapping class group

$$PSL(2,\mathbb{Z}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad-bc=1 \Big/ \{-1\}$$

acting by $\tau \to \frac{a\tau+b}{c\tau+d}$ $\quad d^2\tau \to \frac{d^2\tau}{|c\tau+d|^4}$

$$\tau_2 \to \frac{\tau_2}{|c\tau+d|^2}$$

and under which generators

$$S: \ \tau \mapsto -\frac{1}{\tau} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$T: \ \tau \mapsto \tau+1 \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S^2 = (TS)^3 = -1 = 1$$

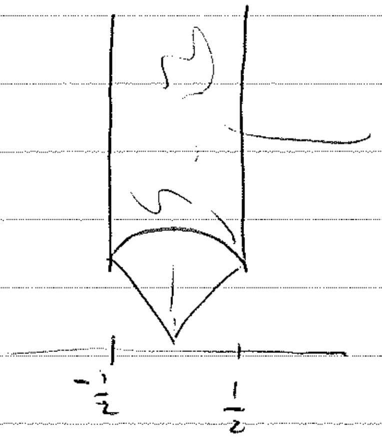
η transforms as

$$\eta(\tau+1) = e^{2\pi i/24}\,\eta(\tau) \qquad \text{(easy to see)}$$

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\;\eta(\tau) \qquad \text{(harder)}$$

As a consequence, Z_1 descends to a well-defined integral over $\mathcal{M}_1 = \mathbb{H}/SL(2,\mathbb{Z})$ or equivalently any fundamental domain such as

$$|\tau| \geq 1 \qquad \mathrm{Re}\,\tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$



$$\int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} = \frac{\pi}{3}$$

<u>Note:</u> $\frac{d^2\tau}{\tau_2^2}$ is the "natural", cause hyperbolic and $SL(2,\mathbb{Z})$-invariant measure on $\mathbb{H}$. It arises in our language from unit volume version of $\hat{h}$

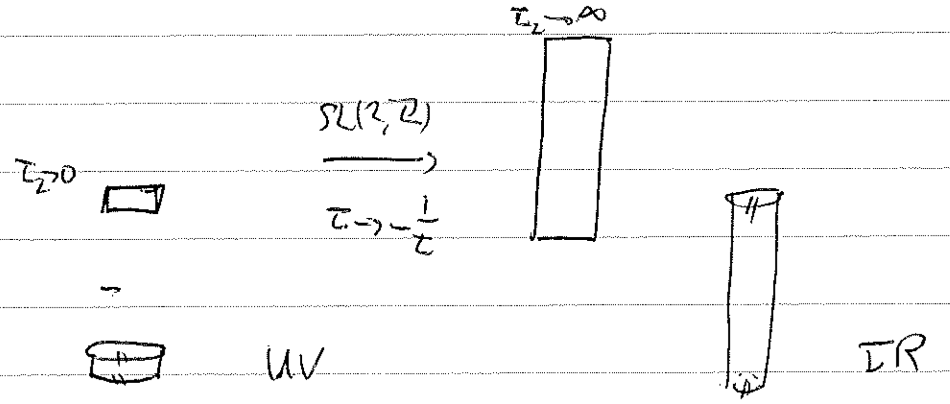
$$\frac{1}{\mathrm{Im}\,\tau}\begin{pmatrix} 1 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & |\tau|^2 \end{pmatrix}$$

⑤ Interpretation

- the ghost contribution (Faddeev-Popov determinant) removes exactly two sets of (light-cone) oscillators and zero modes under the integral sign.
- integration over τ_1 projects (in large τ_2, small q expansion) onto terms ~~with~~ $q^{N^\perp}\bar{q}^{\tilde{N}^\perp}$ with $N^\perp = \tilde{N}^\perp$ (level matching).
- as a consequence, exactly only the physical states contribute to the ~~partition~~ one loop amplitude.
- asymptotic expansion of integrand for $\tau_2 \to 0$ contains information about high (worldsheet) energy density of states $\ln \rho \sim \sqrt{E} \sim M \rightsquigarrow$ Hagedorn temperature

$$T = \frac{1}{4\pi\sqrt{\alpha'}}$$

- From the point of view string = ∞ number of massive particles with $J \sim M^2\alpha'$ however the main point is that the UV region $\tau_2 \to 0$ is completely absent from the integral! Or, if one wishes, mapped to $\tau_2 \to \infty$ (IR) by $SL(2,\mathbb{Z})$



6) Loose ends

- modular invariance is an important consistency condition on the string spectrum and all related calculations.
- calculation of scattering amplitudes involves matter correlators evaluating to various combinations of ϑ-functions.

- the finiteness / all boundaries are spacetime UV regions is very strongly expected to hold in general. But details are subtle.

- For example, a study of open string one-loop diagrams reveals:
  - open strings are not consistent / unitary by themselves. closed strings appear in intermediate channels and need to be included as asymptotic states for unitarity
  - open strings also are not consistent unless they live in unoriented theory

  - this is related to compactness of MCG of disc with one bulk insertion (closed string tadpole) that needs to be cancelled by a corresponding $\mathbb{RP}^2$ diagram in order to ensure one-loop finiteness

    $\rightarrow$ tadpole cancellation & fixing of gauge group.