Winter Semester 2021/22

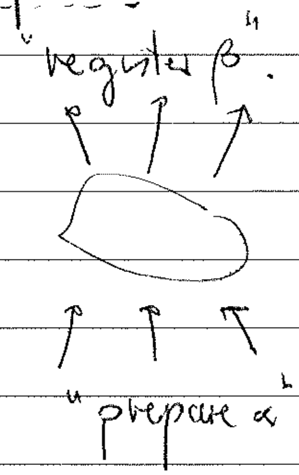
String Theory

Lecture 1 (What, why, how and all that).

WHAT? a relativistic quantum-mechanical (physical) theory of massless fundamental strings.

quantum-mechanical (Observers make) probabilistic predictions about outcome of experiments using Hilbert space(s) $\mathcal{H} \ni |\psi\rangle$ for physical states and Hamiltonian(s) H for time evolution (plural b/c $\mathcal{H}, H$ belong to observer).

Example, scattering experiment



Theory

$$\lim_{t \to -\infty} e^{-iHt/\hbar} |\alpha; in\rangle = |\alpha\rangle$$

$$\lim_{t \to \infty} e^{-iHt/\hbar} |\beta; out\rangle = |\beta\rangle$$

$$\text{Probability}(\beta \leftarrow \alpha) = |\underbrace{\langle \beta; \text{out} | \alpha; \text{in} \rangle}_{(S_{\beta\alpha}) = S\text{-matrix.}}|^2 \boxed{\geq 0.}$$

"unitarity"

relativistic: different (inertial) observers compare their respective experiments & theories, $\mathcal{K}$, H, S with the help of coordinates x^μ and coordinate transformations $x^\mu \to x'^\mu$, (locally ~~...~~ making up) Poincaré group (Lorentz + translations) + general relativistic deformations.

Most of this course:

$$\mu = 0, 1, \dots d$$

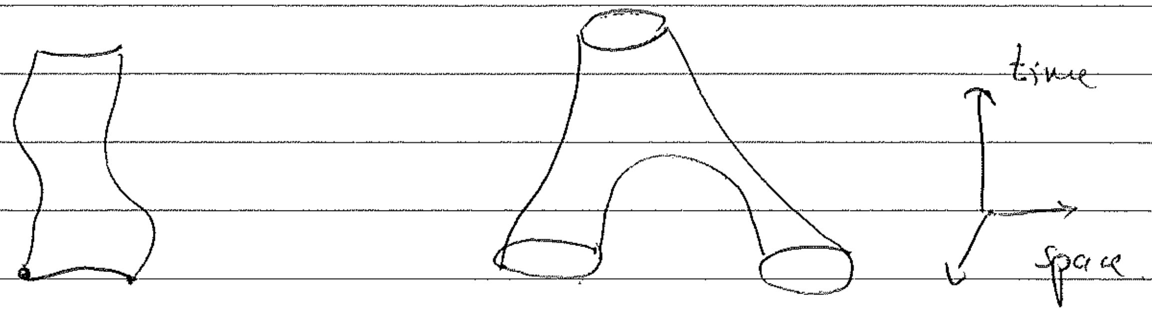
Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1)$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad (\text{E.S.C.})$$

fundamental strings: Q.M. theory arises by quantizing classical dynamics of (compact) 1-dim. objects whose only characteristic ~~is~~ is tension T - dimensionful quantity of dimension E^2. equivalently - ~~length (or time) scale l_s, of~~ area scale α', square of length/time scale l_s.

$$\boxed{T = \frac{1}{2\pi\alpha'} \qquad \alpha' = l_s^2}$$

~> there are open & closed strings (oriented / unoriented) sweeping out worldsheet as the propagate and interact through space & time.



Action principle: relativistic area, $\sim \int_{\Sigma} \det \partial_\alpha X^\mu \partial_\beta X_\mu \, d\sigma^2$

$S = T \cdot A$

allows "ensuring" other principles are satisfied."

This course: explain details, work out consequences.

Note: After the fact, some principles are not (readily) satisfied, and string theory ends up being a lot more than just a "theory of strings" (It is, however, always Q.M. & relativistic.)

in Minkowski space".

"massless": action / mass $= 0$ for pointlike string.

WHY 1? ("why in principle")

1. Quantum gravity

Fact: Dynamics of gravity as described by G.R. is notoriously hard to quantize using traditional methods.
(~~Let me~~ Back of the envelope intuition)

(i) perturbatively

start from: $$S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, R + \text{matter}$$

$$[g] = L^{-2} = M^{-2}$$

$$[R] = L^{-2}$$

Realize: $\rightsquigarrow$ $[G_N] = M^{-2} = L^{2}.$

$$G_N = \frac{1}{M_{Pl}^{2}} \qquad \cancel{G_N = \frac{\hbar c}{8\pi M_{Pl}^2} = \frac{1}{8\pi M_{Pl}^2}}$$

$$M_{Pl} = 1.2 \cdot 10^{19}\ \text{GeV}$$

G_N is dimensionful. $\hbar = c = 1.$

writing $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{Pl}} h_{\mu\nu}$

we obtain

$$S = \int d^4x \, (\partial h)^2 + \frac{1}{M_{Pl}} h (\partial h)^2 + \frac{1}{M_{Pl}^2} h^2 (\partial h)^2 + \dots$$

$$+ \int d^4x \, \frac{1}{M_{Pl}} h_{\mu\nu} T^{\mu\nu}$$

$$T_{\mu\nu} \sim \partial_\mu \phi \, \partial_\nu \phi \dots$$

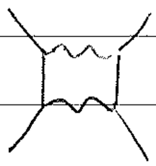
~~and~~ if we try to quantize as in QFT, we obtain a <u>non-renormalizable</u> theory, (infinitely many counterterms and unknown coupling constants).

<u>e.g.</u>

graviton

$$\sim \int^{\Lambda} \frac{d^4k}{k^4} \frac{k^2}{M_{Pl}^2} \sim \frac{\Lambda^2}{M_{Pl}^2}$$

(in fact, Einstein gravity has no divergence at one-loop.)



$$\sim \int \frac{d^4k}{k^8} \frac{k^8}{M_{Pl}^4} \sim \frac{\Lambda^4}{M_{Pl}^4}$$

"unknown physics at the Planck scale"

(ii) "non-perturbatively" (Einstein v. Heisenberg)

- most famous prediction of Einstein gravity: black holes

Concentration of mass m in a region of size $R < R_{Schwarzschild} = G_N \cdot m$ leads to formation of a black hole.

- on the other hand, according Heisenberg, to probe length scales l, we need energy $E \sim \frac{1}{l}$.

Identifying (crudely $l \sim R$ $m \sim E$), we see that probing length scales of size

$$l < G_N \frac{1}{l} \quad : \quad l < \sqrt{G_N} = l_{Planck} = 10^{-33} \text{ cm}$$

leads to formation of black holes $\rightsquigarrow$ ~~no~~ no information.

Moreover, ~~$R_S \sim G_N \cdot E \sim \frac{1}{l}$ in fact grows~~

$$R_S \sim G_N \cdot E \sim \frac{G_N}{l} \quad \text{in fact grows with}$$

energy, inversely to distance probed.

(UV / IR connection).

<u>Conclusion:</u> Theory and experiment will break down at the Planck scale.

→ who cares? Physicists never give up!

<u>Facts about string theory</u>

quantum of h

- $\mathcal{H}_{string} \supset \mathcal{H}_{graviton}$
- (tree-level, short, low-energy) interactions of "gravitons" equal to those in string theory.
- loop amplitudes are finite (no divergences).

<u>Therefore:</u> by identifying $l_s \sim l_{Planck}$, S.T. becomes a candidate for a quantum theory of gravity.

<u>NB.</u> (Perturbative) string theory also does not say what spacetime looks like at the Planck scale.

(~~fundamentally, the~~ if all there is are strings, spacetime should <u>not</u> be a manifold, made of points.)

<u>However:</u>

However:

I String theory respects local Lorentz-invariance for as long as it makes sense to talk about spacetime, whereas many other approaches to Q.G. (loops in particular) break it by discrete structure and it's not clear how to recover at large scales. Experimentally there is compelling evidence (astro-particle physics) that Lorentz-invariance holds up to very high scale, certainly much larger than TeV.

II The (completion of) string theory alluded to above realizes the holographic principle

$$S_{BH} = \frac{A}{4\, l_{Pl}^2} \implies \text{the entropy / information / dofs}$$

Explained in S.T. '96

in q.g. scale with ~~area~~ surface area of spacetime region, instead of volume.

$\Rightarrow$ Any approach to QG based on local QFT is DOOMED.

[Recent developments arising from string theory give evidence for resolution of BH information paradox.]

Ref: Bousso, "the holographic principle"

2 Particle physics

- Standard model of particle physics $SU(3) \times SU(2) \times U(1)$ gauge theory, 3 generations chiral of quarks + leptons, Higgs mechanism, is extremely well tested, but
  - (i) baroque
  - (ii) incomplete (dark matter, dark energy, inflation)
  - (iii) compatible with "grand unification" at $M_{GUT} \sim 10^{16}$ GeV.

- In string theory: $\mathcal{H}^{\text{open}}_{\text{string}} \supset \mathcal{H}_{\text{photon/gluon/}}$

  - $\mathcal{H}_{\text{open string}} \supset \mathcal{H}_{\text{gauge bosons}}$
  - Superstring $\rightarrow$ chiral fermions.
  - Higgs, GUT, inflation, D.M. lots of room.

$\rightarrow$ Simple string models give rough draft of particle physics
Hard to get second draft, but

Still: compelling feature. No other theory on the market for unification.

Ref: Hebecker's book
"Naturalness, string landscape and multiverse"

Of course, ~~would have~~ need of this to worry us if the theory works in practice!

<u>Problem:</u> No direct experimental evidence for either strings, supersymmetry, or any other solid clue to latch on to. <u>So:</u>

<u>WHY2?</u> ("why in practice").

- You might (or might not) choose to not pursue S.T. professionally, perhaps look for & find alternative
  → good to know what is arguably the most successful attempt so far.

- As you might have guessed & we'll see, S.T. is a wonderful playground for testing one's understanding of QM, SRT, QFT, GR, classical mechanics, math etc. ~~→~~
  → useful for other fields. (~~...~~

- S.T. has provided ideas & guidance in other areas: model building, QCD, heavy ions, condensed matter (mostly through AdS/CFT in last 2 decades).
  → Applications perhaps even quantum information & machine learning).

- Tremendous interaction w/ Math, mostly through Susy.
  → what I'm doing.

HOW? (Learning objectives of this class)

→ I will follow the canonical approach to string theory.

1. The relativistic string and its canonical quantization (spectrum and critical dimension)
~ 3-4 weeks

2. The Polyakov path integral, conformal field theory and string interactions
~ 3-4 weeks

3. The superstring, spacetime considerations, supergravity
~ 3-4 weeks

4. String compactifications and string dualities.

This approach is treated in many/any of the many excellent textbooks, including
- Green-Schwarz-Witten
- Polchinski
- Blumenhagen-Lüst-Theisen

and lecture notes, of which ~~two~~
- Polchinski "Little book of strings".
- David Tong
- Timo Weigand.

Lecture 2 (relativistic actions)

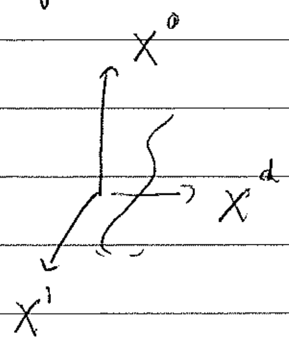
Recall medium-term goal: define Hilbert space + observables "corresponding" to quantization of string dynamics, which to ensure principles and flexibility, should arise from variational principle.

Goal today. Write down such actions.

Traditional starting point: relativistic point particle

1. The relativistic particle

- Consider (free, massive) particle of mass m, moving through Minkowski space $X^0, \dots, X^d$



Question: which trajectories are "physical"?

Answer: straight ~~lines~~ timelike lines

"shortest possible paths", where length is measured in proper time

$$\boxed{L = \int ds}$$

To write formulas, consider on parametrized timelike paths

$(X^\mu(\tau))_{\mu=0,\dots d}$ $\tau \in [\tau_i, \tau_f]$

the action functional

$$S[X(\cdot)] = l = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\dot{X}^2} \qquad \dot{X}^2 =$$

$$\dot{X} = \frac{d}{d\tau} X \qquad \dot{X}^2 = -\dot{X}^{0^2} + \dot{X}^{1^2} + \dots + \dot{X}^{d^2} < 0$$

$$L = -m\sqrt{-\dot{X}^2} \qquad \text{(Lagrangian)}$$

E-L: $$\frac{\delta S}{\delta X^\mu} = -\frac{d}{d\tau} \frac{\partial L}{\partial \dot{X}^\mu}$$

$$P_\mu = m \frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}$$

to recognize this as a straight line, we parametrize using proper time.

$$\tau = \tau(s) \text{ st. } \quad (X')^2 = \left(\frac{d\tau}{ds}\right)^2 \dot{X}^2 = -1$$

const.

$$s = \int \sqrt{-\dot{X}^2}\, d\tau \qquad \text{(always possible \& invertible)}$$

$$\frac{d}{d\tau} = \sqrt{-\dot{X}^2} \frac{d}{ds}$$

$$X'^\mu = \frac{d}{ds}$$

→ The eoms simplify

$$0 = \sqrt{-\dot{X}^2}\,\frac{d}{ds}\,\frac{\sqrt{-\dot{X}^2}}{\sqrt{-\dot{X}^2}}\,X'^\mu = 0$$

i.e. $X''^\mu = 0$

---

features / drawbacks / alternatives

1) reparametrization is a "gauge degree of freedom" (cmp. E-M).
Action is reparametrization invariant. (check this)
Physical trajectories independent of parametrization.

2) It does not make sense to fix gauge before the variation. (at least naively).

NB: This is not a general statement. e.g. in static gauge: $\tau = X^0 = t$

(timelike ⇒ $\dot{X}^0 > 0$).

$$S' = -m \int dt \sqrt{1 - \dot{\vec{X}}^2} = \int m + \frac{1}{2}\dot{\vec{X}}^2 + \dots$$

best for N.R. limit.

drawback: non covariant, (not manifestly Lorentz-invariant.)

$$\frac{\delta S}{\delta x} = \frac{d}{dt}\frac{\dot{\vec{x}}}{\sqrt{1-\dot{\vec{x}}^2}} \rightsquigarrow \frac{d}{dt}\sqrt{1-\dot{\vec{x}}^2} = 0$$

$$\Rightarrow \quad \ddot{\vec{x}} = 0.$$

(which gauges have "good"/"bad" properties is hard to predict & to me one of the most annoying aspect of the business).

3) Action ~~does not make~~ can be generalized to curved spacetime

$$\int \sqrt{-\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}\, d\tau$$

$\rightsquigarrow$ geodesic equation

and also ~~describes~~ allows coupling to E-M field

$$S \to S + \int A_\mu \dot{x}^\mu d\tau.$$

4) does not make sense for massless particles (whether or not we gauge fix.)

5) ~~squareroot makes~~ not very convenient for quantization

P.I. $\rightsquigarrow$ squareroot awkward

Canonical $\rightsquigarrow$ constraints

(more later)

· From many of these pts of view more convenient, although at first sight not, is to replace

"parametrized timelike path"

⇝ one-dimensional embedded subm[anifold] (w/ data of embedding) and "auxiliary" Riemannian metric.

$h = e^2 d\tau^2 \qquad h_{\tau\tau} = e^2 > 0$ and action

$$S(X(\cdot), h) = \frac{1}{2} \int d\mathrm{vol}_e \left(h^{-1}(dX^\mu, dX_\mu) - m \right)$$

"worldline "cosmological constant"

$$= \frac{1}{2} \int d\tau \; e \cdot \left(e^{-2}\, \partial_\tau X^\mu \cdot \partial_\tau X_\mu - m \right)$$

$$S(X(\cdot), e) = \frac{1}{2} \int d\tau \left(\frac{\dot{X}^2}{e} - e m^2 \right)$$

E-L for e: $\quad -\frac{\dot{X}^2}{e^2} = m^2 \qquad e = \frac{\sqrt{-\dot{X}^2}}{m}$

for $\dot{X}$ $\quad \frac{d}{d\tau} \frac{\dot{X}}{e} = \frac{d}{d\tau} \frac{\dot{X}}{\sqrt{-\dot{X}^2}} = 0 \quad$ (same as before!)

Features

1) also reparametrization invariant (w/ understanding that if $\tilde{\tau} = f(\tau)$ $\quad d\tilde{\tau} = f'(\tau)\,d\tau$

then $\tilde{e} = \dfrac{e}{f'(\tau)}$ s.t. $e\,d\tau = \tilde{e}\,d\tilde{\tau}$ ("einbein").

b/c $\quad \partial_\tau = f'(\tau)\,\partial_{\tilde{\tau}}$

$$\rightsquigarrow \; d\tau \frac{(\partial_\tau X)^2}{e} = \frac{d\tilde{\tau}\,(\partial_{\tilde{\tau}} X)^2}{f'\,e} f'^2$$

$$= \frac{d\tilde{\tau}\,(\partial_{\tilde{\tau}} X)^2}{\tilde{e}}$$

2) works smoothly for massless (~~and~~ $m^2 = 0$) and also tachyonic ($m^2 < 0$) particles.

3) We may use reparametrization invariance to fix $\tilde{e} = 1$, whence

$$S = \frac{1}{2}\int d\tilde{\tau}\,\left(X'^2 - m^2\right)$$

with eom

$$X'' = 0.$$

but again, this is not equivalent to original unless we impose by hand the constraint.

$$X'^2 = -m^2$$

- In differential geometry, when one imagines non-relativistic particles moving on Riemannian mf, $\tilde{\tau} = t$ is "physical time", $S = E$ is called Energy (misnomer!) while m and the constraint are ignored. This is fine, as particles will move on geodesic at any (constant) velocity ~~as~~ independent of mass.

- On a pseudo-Riemannian manifold, this is not fully true anymore. ~~It~~ is Precise value of m^2 is irrelevant, but not its sign, constraint is essential,
- Constraint is also essential if one wants $\tilde{\tau}$ = proper time for time like ~~particle~~ motion
- For massless particle, $\tilde{\tau}$ in which $\dot{X}'' = 0$, $X'^2 = 0$ is called "affine parameter", has ambiguous overall scale.

Upshot: There is a lot of fine print in exact ~~obe~~ and complete discussion of any action principle. More in canonical formalism.

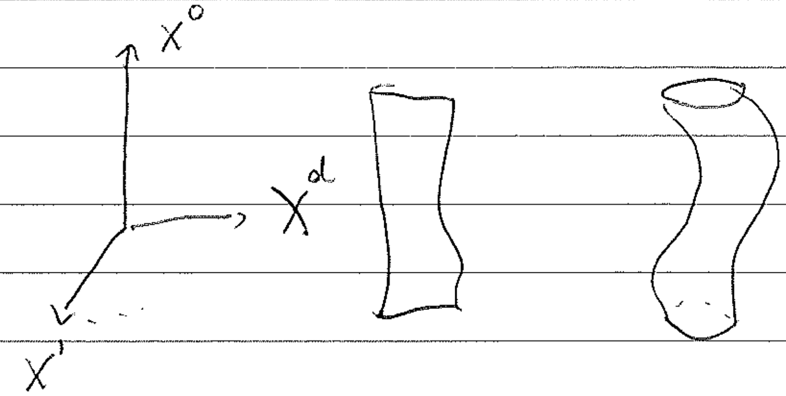
Overview

Principle	manifest Lorentz invariance	local reparametrization invariance	auxiliary data
"proper time" $\int \sqrt{-\dot{X}^2}\, d\tau$	✓	✓	×
"static gauge" $\int dt \sqrt{1 - \dot{\vec{X}}^2}$	×	×	×
"Energy" $\frac{1}{2}\int \frac{d\tau}{e}\left(\dot{X}^2 - e^2 m^2\right)$	✓	✓	1-d metric e
fiducial gauge $\frac{1}{2}\int d\tau\, \dot{X}^2$	✓	×	$\dot{X}^2 = -m^2$ constraint

Beware: It is useful to have these in mind, as we move to string, but again there are important (devilish) differences in the details.

String analogues

oo (open or closed) string moving through Minkowski space



mass m $\longrightarrow$ Tension T

proper time $\longrightarrow$ proper area, ~~…~~ area measured w.r.t. ~~… metric~~ Minkowski metric.

Riemannian situation:

$$dA = |e_1|\,|e_2| \cdot |\sin(\sphericalangle(e_1, e_2))|$$
$$= \sqrt{\left|\,|e_1|^2 \cdot |e_2|^2 - (e_1 \cdot e_2)^2\right|}$$
$$= \sqrt{\det\begin{pmatrix} e_1^2 & e_1 e_2 \\ e_1 \cdot e_2 & e_2^2 \end{pmatrix}}$$

or invariantly, choosing local parametrization τ, σ:

$$A = \int d\tau\, d\sigma \sqrt{\left|\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2\right|}$$

where $\dot{X}^\mu = \frac{d}{d\tau} X^\mu$

$$\dot{X} \cdot X' = \frac{dX^\mu}{d\tau} \cdot \frac{d}{d\sigma} X_\mu \qquad \text{etc.}$$

$$= \int d\tau d\sigma \sqrt{\left| \det \begin{pmatrix} \partial_\tau X^\mu \partial_\tau X_\mu & \partial_\tau X^\mu \partial_\sigma X_\mu \\ \partial_\sigma X^\mu \partial_\tau X_\mu & \partial_\sigma X^\mu \partial_\sigma X_\mu \end{pmatrix} \right|}$$

$$= \int d^2\sigma \sqrt{\left| \det \left(\partial_\alpha X^\mu \partial_\beta X_\mu \right)_{\alpha,\beta = 0,1} \right|}$$

where $(\tau, \sigma) = (\sigma^0, \sigma^1)$

- the action $S_{NG} = -T A$ is known as Nambu-Goto action
- Parameter space (τ, σ) is known as string worldsheet.
- "Physical" string motion - assume there is one timelike and one spacelike tangent vector at each point (actually for reasons to be seen, have to allow light like vector at isolated points - masslessness) $\Rightarrow$ $\underline{\det \leq 0}$

$$\rightsquigarrow S_{N-G} = -T \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} d\tau d\sigma \underbrace{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}}_{\mathcal{L}}.$$

Symmetries: - manifestly Lorentz-invariant

- reparametrisation invariant

$$d^2\sigma \sqrt{|\det \partial_\sigma X \cdot \partial_\sigma X|} = d^2\tilde{\sigma} \sqrt{\left|\det \left(\frac{d\sigma}{d\tilde{\sigma}}\right)^T \partial_\sigma X \partial_\sigma X \frac{d\sigma}{d\tilde{\sigma}}\right|}$$

$$\| \qquad\qquad = d^2\tilde{\sigma} \sqrt{|\det \partial_{\tilde{\sigma}} X \partial_{\tilde{\sigma}} X|}$$

$$d\tilde{\sigma} \left|\det \frac{d\sigma}{d\tilde{\sigma}}\right|$$

$$\| $$

$$\sqrt{\left(\det \frac{d\sigma}{d\tilde{\sigma}}\right)^2}$$

$$\mathcal{L} = -T\sqrt{\quad}$$

→ can be used to simplify eom, after they are derived.
Following usual procedure, define

$$\mathcal{P}^{\tau}_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = -T \frac{X'_{\mu}\, \dot{X}X' - X'^{2}\dot{X}_{\mu}}{\sqrt{(\dot{X}X')^{2} - \dot{X}^{2}X'^{2}}}$$

"canonical momentum"

$$\mathcal{P}^{\sigma}_{\mu} = \frac{\partial \mathcal{L}}{\partial X'^{\mu}} = -T \frac{\dot{X}_{\mu}\, \dot{X}X' - \dot{X}^{2}X'_{\mu}}{\sqrt{(\dot{X}X')^{2} - \dot{X}^{2}X'^{2}}}$$

~~spat~~ worldsheet component of spacetime momentum current.

Then eoms are (in the bulk)

$$\partial_{\tau}\mathcal{P}^{\tau}_{\mu} + \partial_{\sigma}\mathcal{P}^{\sigma}_{\mu} = 0.$$

boundary term (spatial).

$$\int_{\tau_i}^{\tau_f} d\tau \; \mathcal{P}^{\sigma}_{\mu}\,\delta X^{\mu}\Big|_{0}^{\sigma_1} = 0$$

- closed string $\delta X(\sigma_1) = \delta X(0)$

various combinations conceivable - more later
- open string fixed (Dirichlet) $\delta X_1 = 0 \quad \delta X(0) = 0$
- open string free (Neumann) $\mathcal{P}^{\sigma}_{\mu}(\sigma_1) = 0$

$0 = 0$

–

These eqns look complicated, but can be simplified by choosing appropriate gauge, leading to various physical insights into relativistic string dynamics. See e.g. textbook by Zwiebach.

A convenient first step is to fix time reparametrization by choosing "static gauge"

$$\tau = X^0 .\qquad \text{(possible for physical string motion)}$$

Then, without going into full details, consider N-G action:

$$X = (\tau, \vec{x}(\tau,\sigma)) \qquad \dot{X} = (1, \dot{\vec{x}}) \qquad X' = (0, \vec{x}')$$

$$\dot{X}\cdot X' = \dot{\vec{x}}\cdot\vec{x}' \qquad \dot{X}^2 = -1 + \dot{\vec{x}}^2 \qquad X'^2 = \vec{x}'^2$$

$$\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2} = \sqrt{(\dot{\vec{x}}\cdot\vec{x}')^2 + \vec{x}'^2 - \dot{\vec{x}}^2\vec{x}'^2}$$

$$= |\vec{x}'|\sqrt{1 - \underbrace{\left(\dot{\vec{x}}^2 - \frac{(\dot{\vec{x}}\cdot\vec{x}')^2}{\vec{x}'^2}\right)}}$$

$$= \left(\dot{\vec{x}} - \frac{\dot{\vec{x}}\cdot\vec{x}'}{\vec{x}'^2}\,\vec{x}'\right)^2 = \vec{v}_\perp^{\,2}$$

$$S = -T\int d\tau \int d\sigma \underbrace{\frac{ds}{d\sigma}}_{=\,\vec{x}'} \sqrt{1 - \vec{v}_\perp^{\,2}}$$

$$= -T\int d\tau \int ds \sqrt{1 - \vec{v}_\perp^{\,2}}$$

$\leadsto$ string element of length ds $\longleftrightarrow$ particle of mass $dm = T\,ds$ moving at velocity $\vec{v}_\perp$

T = mass density, longitudinal modes are pure gauge.

21a

Instead of doing the rest by foot, let's take the bicycle.

Recognize $\quad \gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$: induced metric

use formula from L.A.

$$\delta \det M = \det M \; \mathrm{tr}\, M^{-1} \delta M$$
$$= \mathrm{tr}\,(M^{ad}\, \delta M)$$

adjunct matrix

and the following important theorem from diff. geom.:

[Any 2-d metric is locally conformally flat.]

i.e. $\exists$ local coordinates in which

$$\gamma_{\alpha\beta} = \underbrace{\text{scale factor}}\cdot \underbrace{\text{fiducial (const.) metric}}$$

$$= \sqrt{-\det \gamma_{\alpha\beta}} = \sqrt{-\gamma} \qquad \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Minkowski signature}$$

$$= \sqrt{\gamma} \qquad \delta_{\alpha\beta} \quad \text{~~Riemannian~~ euclidean signature}$$

Then

$$S_{NG} = -T \int d^2\sigma \sqrt{-\gamma} \qquad \gamma = \det \gamma_{\alpha\beta}, \; \gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$$

$$\delta S = \partial_\alpha \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \partial_\beta X_\mu$$

fiducial gauge: $\sqrt{-\gamma}\, \gamma^{\alpha\beta} = \eta^{\alpha\beta}$

so

So eom is simply

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X_\mu = -\ddot{X}_\mu + X''_\mu = 0$$

(2d wave equation)

supplemented by constraints

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \gamma \cdot \eta_{\alpha\beta}\,.$$

$$\dot{X} \cdot X' = 0$$

"as simple as it gets"

$$\dot{X}^2 - X'^2 = 0$$