SUPERGEOMETRY AND SUPERGRAVITY HW 4

HEIDELBERG UNIVERSITY 2017/18

1. Cartan calculus

***Problem* 1** (Chern-Simons Forms)**:** Consider a principal connection on a principal $G = GL(r)$-bundle over a manifold M, locally represented by a connection one-form $A \in \Omega^1(U, \mathfrak{g})$. Verify by explicit computation that

$$
\begin{aligned}
dC_3 &= \mathrm{tr}\big(F \wedge F\big) \\
dC_5 &= \mathrm{tr}\big(F \wedge F \wedge F\big)
\end{aligned}
\tag{1}
$$

where $F = dA + \frac{1}{2}[A \wedge A]$ is the curvature form and

$$
\begin{aligned}
C_3 &= \mathrm{tr}\big(F \wedge A - \frac{1}{3} A \wedge A \wedge A\big) \\
C_5 &= \mathrm{tr}\big(F \wedge F \wedge A - \frac{1}{2} F \wedge A \wedge A \wedge A + \frac{1}{10} A \wedge A \wedge A \wedge A \wedge A\big)
\end{aligned}
\tag{2}
$$

Also show (explicitly) that C_3 and C_5 change by an exact form under infinitesimal gauge transformations $\delta A = d\lambda + [A \wedge \lambda]$ (for some local function $\lambda \in \Omega^0(U, \mathfrak{g})$).

***Problem* 2** (The First Order formalism)**:** We showed in class that the Palatini action

$$
S_P = \int_M \epsilon\big(\Omega \wedge \theta \wedge \cdots \wedge \theta\big) \tag{3}
$$

for vielbein θ plus connection (one-form) ω is equivalent to ordinary Einstein gravity. Here, $\epsilon : \otimes^n TM \to \mathbb{R}$ is the $SO(n)$ invariant projection (determinant) factoring through the isomorphism $\wedge^n TM \cong \mathbb{R}$ otherwise known as Riemannian volume form.

When $n = 4$, $\otimes^4 TM$ contains an alternative scalar invariant that can be described as the inner product $\eta : \wedge^2 TM \times \wedge^2 TM \to \mathbb{R}$ induced by the pseudo-Riemannian metric g on M. Evaluate the Immirzi action

$$
S_I = \int_M \eta\big(\Omega \wedge \theta \wedge \theta\big) \tag{4}
$$

and its variation with respect to θ and ω in terms of contraction of the Riemann and torsion tensor. (With respect to a local trivialization of TM one has simply:

$$
\eta\big(\Omega \wedge \theta \wedge \theta\big) = \Omega_{ab} \wedge \theta^a \wedge \theta^b \quad) \tag{5}
$$

2. Symplectic Geometry

***Problem* 3** (Tautological 1-form)**:** Let X be a smooth n-dimensional manifold and $M = T^*X$ its cotangent bundle. Choose a local coordinate chart $\mathcal{U}$ on X with coordinates $(x_1, \dots x_n)$ where $x_i : \mathcal{U} \to \mathbb{R}$. Use the projection $\pi : T^*X \to X$ to construct a canonical 1-form $\lambda : TM \to \mathbb{R}$ on the cotangent bundle M of X.

If $\xi \in T^*_x X$, then $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$ for some real coefficients ξ_i. The chart $T^*\mathcal{U}$ with coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ is a coordinate chart for M. Write λ in terms of local coordinates.

***Problem* 4** (Lagrangian Submanifolds)**:** If $\phi : TX \to \mathbb{R}$ is a 1-form on a (smooth) manifold X then we can consider it as a mapping from X to T^*X. Show that $\phi(X)$ is a Lagrangian submanifold if and only if ϕ is closed.

It follows that Lagrangian submanifolds of T^*X that project diffeomorphically onto X are in bijection with closed 1-forms on X. If the 1-form corresponding to a Lagrangian submanifold is of the form dS, S is called a *generating function*. Hence we can think of Lagrangian submanifolds of T^*X as "generalized functions" on X.

Let $f : M_2 \to M_1$ be a diffeomorphism. The *graph of f* is by definition

$$\Gamma_f = \{(f(x), x) | x \in M_2\} \subset M_1 \times M_2.$$

***Problem* 5** (Symplectomorphisms)**:** Show that Γ_f is a Lagrangian submanifold of $(M_1, \omega_1) \times (M_2, -\omega_2) = (M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$ if and only if f is a symplectomorphism.

***Problem* 6** (Canonical Transformations)**:** Explain the canonical transformations of type I, II, II, and IV in terms of the previous problem. Fix the discussion at https://en.wikipedia.org/wiki/Canonical_transformation.