SUPERGEOMETRY AND SUPERGRAVITY HW 2

HEIDELBERG UNIVERSITY 2017

***Problem* 1** (The category of (left) G-sets)**:** A left G-action of a group G on a set X is a map

$$(\cdot) : G \times X \to X$$

satisfying

$$(gh) \cdot x \cong g \cdot (h \cdot x) \text{ and } 1 \cdot x = x.$$

The category (left) G-sets has

(a) **Objects**: Sets X with a left G-action $G \times X \to X$.
(b) **Morphisms**: G-equivariant maps $\phi : X \to Y$, e.g. $g \cdot \phi(x) = \phi(g \cdot x)$.

The Cartesian product of two G-sets X and Y is the set $X \times Y$ and G-action

$$g \cdot (x, y) = (g \cdot x, g \cdot y).$$

Let X and Y be two G-sets. Show that the space of inner-Homs between X and Y is the set of all mappings $\xi : X \to Y$ (not necessarily G-homs from X to Y) with the action

$$(g \cdot \xi)(x) = g \cdot \xi(g^{-1} \cdot x).$$

Let $Z^0(\underline{Hom}(X, Y))$ be the space of G-invariant mappings between X and Y. Show that

$$Z^0(\underline{Hom}(X, Y)) = Hom_{G-set}(X, Y).$$

***Problem* 2** (The category $DGVec_{\Bbbk}$)**:** The category of $DGVec_{\Bbbk}$ has:

(a) **Objects**: DG-vector spaces (V, d_V). A DG-vector space is the data of a graded vector space $V = \oplus_{n\in\mathbb{Z}} V^n$ together a linear map $d : V \to V$, called the differential, such that $d(V_n) \subset V_{n+1}$ for every n and $d^2 = d \circ d = 0$.
(b) **Morphisms**: A morphism

$$f : (V, d_V) \to (W, d_W)$$

of DG-vector spaces is a linear map $V \to W$ such that $f(V^n) \subset W^n$ for every n and $d_W f = f d_V$.

Given two DG-vector spaces V and W their tensor product is the DG-vector space

$$V \otimes W = \oplus_{n\in\mathbb{Z}}(V \otimes W)^n = \oplus_{i+j=n} V^i \otimes W^j$$

with differential

$$d(v \otimes w) = dv \otimes w + (-1)^{\overline{v}} v \otimes dw.$$

We can view a DG vector space as a cochain complex

$$\cdots \xrightarrow{d_V} V^n \xrightarrow{d_V} V^{n+1} \xrightarrow{d_V} V^{n+2} \xrightarrow{d_V} \cdots$$

and morphisms of DG vector spaces as cochain maps

$$\begin{tikzcd}
\cdots \arrow[r, "d_V"] & V^n \arrow[r, "d_V"] \arrow[d] & V^{n+1} \arrow[r, "d_V"] \arrow[d] & V^{n+2} \arrow[r, "d_V"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d_W"] & W^n \arrow[r, "d_W"] & W^{n+1} \arrow[r, "d_W"] & W^{n+2} \arrow[r, "d_W"] & \cdots
\end{tikzcd}$$

(a) For DG-vector spaces V and W, determine

$$\underline{Hom}(V, W)$$

by describing its graded components $\underline{Hom}^n_{\Bbbk}(V, W)$ and explicitly describe the differential:

$$d : \underline{Hom}^n_{\Bbbk}(V, W) \to \underline{Hom}^{n+1}_{\Bbbk}(V, W).$$

(b) Show that

$$Z^0(\underline{Hom}(V, W)) \cong Hom_{DGVec_{\Bbbk}}(V, W),$$

where Z^0 denotes the d-closed objects.