SUPERGEOMETRY AND SUPERGRAVITY HW 1

HEIDELBERG UNIVERSITY 2017

***Problem* 1** (Haskell Exercises)**:** Write functions with the following type signatures:

```
id' :: a -> a
eval :: (a -> b, a) -> b
exchange :: (a, b) -> (b, a)
compose :: (b -> c) -> (a -> b) -> a -> c
curry' :: ((a,b) -> c) -> (a -> b -> c)
associate :: (a, (b, c)) -> ((a, b), c)
```

Such that `eval` takes a function f and a value x and returns $f(x)$, `compose` is the compositions of two functions f and g. What is the type signature of the function

```
lift (f , g) (a,b) = (f a, g b)
```

Define the function

```
rpentagon = associate . associate
```

What is its type signature? Define another function `lpentagon` that is a composition of three functions that completes the other side of MacLane's pentagon identity.

***Problem* 2** (Winning the Lottery)**:** For each ticket you buy in the Pick-3 lottery, you can choose a set of three distinct integers from $\{1, 2, 3, 4, 5, 6, 7\}$. A ticket wins a prize if at least two of the numbers chosen on the ticket match three integers (randomly) drawn from $\{1, 2, 3, 4, 5, 6, 7\}$.

(1) What is the minimal number of tickets you must buy to ensure that you have at least one winning ticket?
(2) Give an explicit example of such a minimal set.
(3) Given two minimal winning sets, show that they can be related to each other simply by permuting all the integers.
(4) What is the symmetry group of the set of minimal winning sets? What is its order? Give an explicit description as a subgroup of the symmetry group on seven letters.

***Problem* 3** (Triangulations)**:** Construct a triangulation of the real projective plane $\mathbb{RP}^2$ with the minimal possible number of faces. How many faces are there? What is the symmetry group of the triangulation?

***Problem* 4:** Let $(\mathscr{C}, \otimes)$ be a tensor category with an inner Hom. Define one (or two) morphisms

$$M^\vee \otimes N \to \underline{\mathrm{Hom}}(M, N)$$

and show that for $\mathscr{C} = \mathbf{sVec}$, one (or both) of these morphisms, when applied on an element $\omega \otimes n \in M^\vee \otimes N$, yields the morphism

$$m \mapsto (-1)^{p(\omega)p(n)} n\omega(m).$$

***Problem* 5** (A particle on a circle)**:** Consider a particle on a circle with the Lagrangian

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2\pi}\theta\dot{q}$$

where $q \cong q + 2\pi$.

(1) Determine the Hamiltonian H in terms of the conjugate momentum Π_q.
(2) Explain why $\theta \to \theta + 2\pi$ is a symmetry of the system.
(3) Since H_θ and $H_{\theta+2\pi}$ describe the same physical system, they must be related by a unitary operator $U(q)$. Determine $U(q)$.
(4) Determine the position eigenfunctions $|n\rangle, n \in \mathbb{Z}$, and show that the energy eigenvalues are

$$E_n = \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2.$$

Note that the spectrum is invariant under $\theta \to \theta + 2\pi$. We now determine the symmetries of the system. For any value of θ the system is invariant under the action of the translation operator $\mathcal{O}_\alpha : q \to q + \alpha$. However at $\theta = 0$. These operators form the group $U(1)$. However for $\theta = 0$ and $\theta = \pi$ there is an additional discrete symmetry $\mathcal{O}_C : q \to -q$.

(1) Determine the action of $\mathcal{O}_\alpha$ and $\mathcal{O}_C$ on the eigenstates $|n\rangle$ when $\theta = \pi$.
(2) Determine the action of $\mathcal{O}_C\mathcal{O}_\alpha\mathcal{O}_C$ on $|n\rangle$.
(3) Introduce an operator $I_\beta : |n\rangle \to e^{i\beta}|n\rangle$ and consider $V_\alpha = I_{-\alpha/2}\mathcal{O}_\alpha$ with $\alpha \in [0, 4\pi)$. Determine the action of

$$\mathcal{O}_C V_\alpha \mathcal{O}_C.$$

(4) What is the group G_1 generated by $\mathcal{O}_\alpha$ and $\mathcal{O}_C$?
(5) What is the group G_2 generated by V_α and $\mathcal{O}_C$?
(6) How are these two groups related?