Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 8

Abgabe: Donnerstag, den 10. Dezember 2015,

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1. Non-matrix Lie group, quotient version

Let $G = \{(x, y, u) | x \in \mathbb{R}, y \in \mathbb{R}, u \in U(1)\}$ with product given by

$$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{2\pi i x_1 y_2} u_1 u_2)$$

We will show that G has no faithful finite-dimensional representation. Let H be the Heisenberg group of unitriangular matrices, and consider the map $\Phi : H \to G$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a, b, e^{2\pi i c}).$$

i) Verify that Φ is a homomorphism and determine its kernel.

ii) Determine the center $Z(H)$.

Now let Σ be any finite dimensional representation of G. By composing we get a representation $\Pi = \Sigma \circ \Phi$ of H. The kernel of Π clearly contains the kernel of Φ. We will set out to show that in fact $Z(H)$ is also necessarily in the kernel of Π. Then Σ must kill $\Phi(Z(H))$ and cannot be faithful.

The Lie algebra $\mathfrak{h}$ of H is spanned by elements X, Y, Z satisfying $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. Let π be the representation of $\mathfrak{h}$ associated to Π.

iii) Show that for any two commuting operators A, B on a finite dimensional vector space V, for any eigenvalue λ of A, the operator B takes the eigenspace V_λ of A to itself. Using this, show that the restriction of $\pi(Z)$ to any of its eigenspaces is zero. In other words all the eigenvalues of $\pi(Z)$ are zero, or again $\pi(Z)$ is nilpotent.

iv) Show that if A is any nilpotent matrix and $e^{tA} = I$ for some *non-zero* $t \in \mathbb{R}$, then $A = 0$.

Explicitly the matrices for $\mathfrak{h}$ can be taken to be

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

v) Compute $\exp(Z)$. Put together the evidence to conclude that $Z(H) \subset \ker(\Pi)$, hence Σ is not faithful.

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2. Non-matrix Lie group, cover version

i) Show that $SL(2,\mathbb{R})$ is not simply connected.

Hint: one nice way to visualize this is to parametrize $SL(2,\mathbb{R})$ *as* $\begin{pmatrix} x_1+x_2 & x_3+x_4 \\ x_3-x_4 & x_1-x_2 \end{pmatrix}$ *and determine the quadric in* $\mathbb{R}^4$ *cut out by* $\det = 1$*. This quadric is also known as 3-dimensional Anti-de Sitter space.*

ii) Show that $SL(2,\mathbb{C})$ is simply connected.

Hint: consider the fibration $SL(2,\mathbb{C}) \to \mathbb{C}^2 \setminus \{0,0\} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a,b)$.

We will show that the universal cover of $SL(2,\mathbb{R})$ is not a matrix Lie group. Let $G \subset GL(n,\mathbb{C})$ be a Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Suppose $\Phi : G \to SL(2,\mathbb{R})$ is a Lie group homomorphism for which the differential is an isomorphism of $\mathfrak{sl}(2,\mathbb{R})$ to itself. We claim that Φ is then necessarily a Lie group isomorphism, so that G cannot be the universal of $SL(2,\mathbb{C})$.

iii) Suppose $\psi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{C})$ is a Lie algebra homomorphism. Show that there exists a Lie group homomorphism $\Psi : SL(2,\mathbb{R}) \to GL(n,\mathbb{C})$ such that $\Psi(e^X) = e^{\psi(X)}$ for all $X \in \mathfrak{sl}(2,\mathbb{R})$.

Hint: First establish the analogous result for the complex-linear extension $\psi_{\mathbb{C}} : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$.

iv) Now let ψ be the inverse of the Lie algebra isomorphism ϕ above. Show that the map Ψ whose existence you just established and the Lie group homomorphism Φ that ϕ came from must be inverses of each other. Conclude that G can not be the universal cover of $SL(2,\mathbb{C})$.

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