Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 5

Abgabe: Donnerstag, den 19. November 2015

Coordinates on $SU(2)$

Recall the homomorphism $SU(2) \to SO(3)$: If we identify $\mathbb{R}^3$ with the space of 2×2 traceless hermitian matrices (up to a factor of i, the Lie algebra $su(2)$, or the imaginary quaternions) via

$$\mathbb{R}^3 \ni (x, y, z) \mapsto a = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix} \in su(2)$$

(where the Pauli matrices σ_x, σ_y, σ_z give a basis for $su(2)$), then, for $g \in SU(2)$ the linear map $\rho(g)$ defined by

$$\rho(g)(a) = gag^\dagger$$

preserves the standard Euclidean inner product and the orientation on $\mathbb{R}^3$, and therefore $\rho(g) \in SO(3)$.

1. (i) Show that under this homomorphism, the diagonal matrix

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \in SU(2)$$

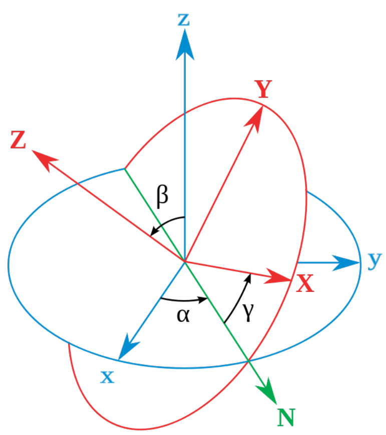
maps to a rotation $R_z(\alpha)$ by the angle α around the z-axis.

(ii) Find similarly the preimage in $SU(2)$ of the rotations R_x, R_y around x- and y-axis, respectively.

According to L. Euler, any rotation in three-dimensional space can be represented as the sequence of three rotations

$$R_z(\gamma) \circ R_x(\beta) \circ R_z(\alpha)$$

(The figure to the right will help you remember...)



2. Using the above homomorphism $SU(2) \to SO(3)$, lift the Euler angles to a coordinate system on $SU(2)$. What is the maximal range of the angles α, β, γ ?

Please turn over

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Hint: You may find it convenient to recall the identification $SU(2) \cong S^3 = \{|z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ given by

$$g = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

and amusing to compare it with the identification $su(2) \cong \mathbb{R}^3$ given above.

Invariant vector fields on $SU(2)$

Let G be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is called *left-invariant* if $(DL_g)_h(X_h) = X_{gh}$ for all $g, h \in G$. Right-invariant vector fields are defined similarly (how exactly?). As you'll know, the vector space of left-invariant vector fields is isomorphic to the tangent space at the identity, *i.e.*, the Lie algebra $\mathfrak{g}$ of G. The goal of the next two problems is to write down these vector fields explicitly in the Euler angles on $SU(2)$.

3. (i) Working (preferably) in real coordinates, $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ on $\mathbb{C}^2 \cong \mathbb{R}^4 \supset SU(2) \ni g$, what is the tangent space $T_g SU(2)$ as a subspace of the tangent space $T_g \mathbb{R}^4$?

   (ii) By extending left-translation L_g to a linear map on $\mathbb{R}^4$, calculate the three vector fields $X_g = (DL_g)_e(\sigma_x)$, $Y_g = (DL_g)_e(\sigma_y)$, $Z_g = (DL_g)_e(\sigma_z)$ at $g \in SU(2)$ in the coordinate basis for the tangent space $T_g \mathbb{R}^4$. Make sure your result is in the subspace $T_g SU(2)$!

   (iii) Calculate the Lie brackets of the vector fields X, Y, Z.

   (iv) (Extra credit) Repeat (ii) and (iii) with L_g replaced with right-translation R_g.

4. Calculate the components of the left-invariant vector fields X, Y, Z in the basis ∂_α, ∂_β, ∂_γ of $T_g SU(2)$ corresponding to the Euler angles found in problem 2.
   You may check your result by verifying the Lie algebra.