Multiplication in S_3

The elements are $\{1, (12), (13), (23), (123), (132)\}$

To compute products, we identify the elements with their action on the set $\{1,2,3\}$, for example as follows:

$$(12): \begin{array}{c} 1\ 2\ 3 \\ \downarrow \\ 2\ 1\ 3 \end{array} \qquad (123): \begin{array}{c} 1\ 2\ 3 \\ \downarrow \\ 2\ 3\ 1 \end{array}$$

To calculate the product $(123)\cdot(12)$, which we take to mean "apply (12), then (123)", we work out the composite operation

$$\begin{array}{c} 1\ 2\ 3 \\ \downarrow (12) \\ 2\ 1\ 3 \\ \downarrow (123) \\ 3\ 2\ 1 \end{array} = \begin{array}{c} 1\ 2\ 3 \\ \Big\downarrow (13) \\ 3\ 2\ 1 \end{array} \qquad \text{i.e. } (123)\cdot(12) = (13)$$

To analyze the regular representation one can start by calculating the product of the elements $\tau = (123)$ and $\sigma = (12)$ with all the others; the result for $\tau = (123)$ is the following:

$\cdot$	1	(12)	(13)	(123)	(132)
(123)	(123)	(13)	(23)	(132)	1

(note that this is easy to deduce without doing any additional calculations.)

Then the matrix representing τ in the regular representation is

$$\rho_R(\tau) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Decomposing $V \otimes V$

The standard representation V of S_3 has a basis in which σ and τ have the following matrix representation:

$$\rho_V(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_V(\tau) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad \text{with } \omega = e^{2\pi i/3}$$

Denoting the basis vectors of V by α, β, a basis for $V \otimes V$ is given by $\alpha\otimes\alpha$, $\alpha\otimes\beta$, $\beta\otimes\alpha$, $\beta\otimes\beta$. It is straightforward to calculate the action of σ and τ in that basis (note $\omega^3 = 1$):

$$\alpha\otimes\alpha \begin{cases} \xrightarrow{\sigma} \beta\otimes\beta \\ \xrightarrow{\tau} \omega^2\, \alpha\otimes\alpha \end{cases} \qquad \alpha\otimes\beta \begin{cases} \xrightarrow{\sigma} \beta\otimes\alpha \\ \xrightarrow{\tau} \alpha\otimes\beta \end{cases} \qquad \beta\otimes\alpha \begin{cases} \xrightarrow{\sigma} \alpha\otimes\beta \\ \xrightarrow{\tau} \beta\otimes\alpha \end{cases} \qquad \beta\otimes\beta \begin{cases} \xrightarrow{\sigma} \alpha\otimes\alpha \\ \xrightarrow{\tau} \omega\, \beta\otimes\beta \end{cases}$$

In other words the matrices are

$$\rho_{V\otimes V}(\sigma) = \begin{pmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & 1 & 0 & \\ 1 & & & 0 \end{pmatrix}, \qquad \rho_{V\otimes V}(\tau) = \begin{pmatrix} \omega^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & \omega \end{pmatrix}$$

Sincerest apoligies.

Upon diagonalizing $\rho_{V\otimes V}(\sigma)$, i.e. trading $\alpha\otimes\beta$ and $\beta\otimes\alpha$ in favor of $\alpha\otimes\beta + \beta\otimes\alpha$ and $\alpha\otimes\beta - \beta\otimes\alpha$ with σ-eigenvalue $+1$ and -1 respectively, we see that $V\otimes V$ is isomorphic to $U \oplus U' \oplus V$ with U and U' the trivial and sign representations.

$$\begin{pmatrix} \rho_U(g) = 1 \quad \forall g \\ \rho_{U'}(g) = \mathrm{sgn}(g) \quad \forall g \end{pmatrix}$$