Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 1

Abgabe: Donnerstag, den 15. Oktober 2015,
im Anschluss **Besprechung** im HS 4, INF 288

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1. Practice with $\mathcal{S}_3$

Fulton & Harris, Exercise 1.11:

In the notation of Fulton & Harris, $V = \langle \alpha, \beta \rangle$ (angled brackets denote the span), where in this basis

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$.

Find the decomposition of $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^3 V$ in irreducible representations.

2. More practice with $\mathcal{S}_3$

Fulton & Harris, Exercise 1.12

The *regular* representation R of a group G has one basis vector for each group element and the action is given in this basis by $\rho_R(g)(v_h) = v_{gh}$.

(i) Decompose the regular representation R of $\mathcal{S}_3$.

(ii) Show that $\mathrm{Sym}^{k+6} V$ is isomorphic to $\mathrm{Sym}^k V \oplus R$, and compute $\mathrm{Sym}^k V$ for all k.

3. Finite subgroups of $SO(3)$

Let a finite group G act on a finite set S. Recall the following definitions of orbit and stabiliser: for any $s \in S$,

$$O_s = \{g \cdot s,\, g \in G\}$$
$$G_s = \{g \in G,\, g \cdot s = s\}$$

(i) Recall the *Counting Formula*

$$|G| = |G_s||O_s|.$$

Show that if $s' \in O_s$, then $|G'_s| = |G_s|$.

Recall the definition of $SO(3)$ the group of rotations about the origin in $\mathbb{R}^3$. We can use the counting formula to classify its finite subgroups. So let now G be a hypothetical finite subgroup of $SO(3)$, of order $n = |G|$.

(ii) Any non-trivial element $g \in G \setminus \{e\}$ is a rotation of finite order around some axis in $\mathbb{R}^3$. Denote the collection of poles of those axes (*i.e.*, their intersections with $S^2 \subset \mathbb{R}^3$) by P (note that each $g \neq e$ yields two poles, but usually there are many g for each pair of poles). Show that G acts on the set of poles P.

(iii) Let $r_p = |G_p|$. Show that

$$\sum_{p \in P} (r_p - 1) = 2n - 2.$$

Label the orbits P_i, $i = 1, \ldots, k$ and let r_i be the order of the stabiliser of one of its elements (by (i), this is the same for any of them). Show that

$$\sum_{i=1}^{k} \left(1 - \frac{1}{r_i}\right) = 2 - \frac{2}{n} \tag{1}$$

(iv) Find all integer solutions to (1). You should find three distinct "families" (two families, and "exceptional" cases). Interpret your results in terms of symmetries of familiar geometric figures (the Platonic solids).

The simple Lie algebras, as we will learn this semester, were classified by Dynkin with diagrams labelled by letters A to F, with the ones labeled A, D and E having a special feature in common. Rather interestingly, many other objects in mathematics are classified according to ADE diagrams. We will see later that the three families you found correspond to A, D, and E respectively.

4. Dual representation

(i) Show that the fundamental (defining, matrix) representation of $SU(2)$ is equivalent to its dual.

(ii) Find an example of a representation that is not equivalent to its dual.