Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 9

Abgabe: Donnerstag, den 17. Dezember 2015,

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1. "A little plethysm" a day keeps the rust away, or the addition of angular momentum

Let V be the fundamental 2-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. Show that for $a \geq b$,

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V = \mathrm{Sym}^{a+b} V \oplus \mathrm{Sym}^{a+b-2} V \oplus \cdots \oplus \mathrm{Sym}^{a-b} V$$

(*Fulton & Harris, exercise 11.11*)

Hint: A good way to keep track of the "spin content" is to multiply out the polynomial $(t^a + t^{a-2} + \cdots + t^{2-a} + t^{-a})(t^b + t^{b-2} + \cdots + t^{2-b} + t^{-b})$.

2. Miscellaneous, from Humphreys

Let F be any field.

i) Let $x \in \mathfrak{gl}(n, F)$ have n distinct eigenvalues a_i. Show that the eigenvalues of $\mathrm{ad}(x)$ are precisely the n^2 scalars $a_i - a_j$, which of course need not be distinct.

ii) Show that $\mathfrak{sl}(n, F)$ is precisely the derived algebra of $\mathfrak{gl}(n, F)$.

iii) Show that the center of $\mathfrak{gl}(n, F)$ is $\mathfrak{s}(n, F)$ (the scalar matrices). Determine the center of $\mathfrak{sl}(n, F)$.

3. Oscillators

The quantum mechanical Hilbert space of the harmonic oscillator can be used to build unitary representations of $SL(2,\mathbb{R})$ and $SU(2)$. We first briefly recall basic definitions.

Consider the algebra of operators $a, a^\dagger$ satisfying $[a, a^\dagger] = 1$. The Hilbert space of the harmonic oscillator is spanned[1] by a lowest weight (or vacuum) vector $|0\rangle$ satisfying $a|0\rangle = 0$, and the infinite tower $|n\rangle = 1/\sqrt{n!}(a^\dagger)^n|0\rangle$ above it of images under $a^\dagger$. [2]

The Hilbert space is equipped with a Hermitian inner product $\langle\cdot|\cdot\rangle$, and the inner product of vectors $|\phi\rangle$ and $|\psi\rangle$ is often denoted $\langle\phi|\psi\rangle$. It is defined by $\langle 0|0\rangle = 1$ and $\langle n|m\rangle = 0$ for $n \neq m$. The normalization of $|n\rangle$ is chosen so that $\langle n|n\rangle = 1$. Check this if you have never done so!

I.) $SL(2,\mathbb{R})$

Define $L_+ = (1/\sqrt{2})a^\dagger a^\dagger$ and $L_- = (1/\sqrt{2})aa$.

i) Calculate the commutator $[L_+, L_-]$ and call it L_0.

ii) Calculate $[L_0, L_\pm]$. You should discover that L_+, L_-, L_0 generate an $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra.

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[1] in the Hilbert space sense, i.e. finite linear combinations of the given vectors are dense in the Hilbert space.

[2] if you prefer, you can also say that the Hilbert space is $\mathcal{H} = l^2(\mathbb{N}_0)$.

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iii) Starting from the vacuum $|0\rangle$, applying L_+ repeatedly we obtain a representation spanned by all the even states $|2n\rangle$, and starting from $|1\rangle$ we obtain one spanned by all the odd ones $|2n+1\rangle$. Compute the L_0 eigenvalue on each and conclude in particular that the two representations are not isomorphic.

iv) Show that while the commutator $[a, a^\dagger]$ is not in the subalgebra generated by L_+, L_-, L_0, their anticommutator $\{a, a^\dagger\}$ is. Adjoining a and $a^\dagger$ thus gives rise to a super Lie algebra, and the two representations above combine into a single representation of this super Lie algebra.

These are the simplest unitary[3] representations of $SL(2,\mathbb{R})$, which are all infinite dimensional.

II.) $SU(2)$

Take now two "uncoupled" copies of the harmonic oscillator algebra, generated by a_+, $a_+^\dagger$ and a_-, $a_-^\dagger$. Define $J_+ = a_+^\dagger a_-$ and $J_- = a_-^\dagger a_+$.

i) Calculate the commutator $[J_+, J_-]$ and call it J_3.

ii) Calculate $[J_3, J_\pm]$. You should discover that J_+, J_-, J_3 generate an (complexified) $\mathfrak{su}(2)$ Lie algebra.

iii) The Hilbert space is spanned by all vectors $|n_+, n_-\rangle$. Show that it is isomorphic to a direct sum of one copy of each finite dimensional irreducible representation of $\mathfrak{su}(2)$.

[3] though we are not explaining this here.