Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 7

Abgabe: Donnerstag, den 3. Dezember 2015,

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1. Surjectivity of the exponential map

Show that the exponential map from $\mathfrak{g} = sl(2,\mathbb{R}) = \{X \in \mathrm{Mat}_2(\mathbb{R}), \mathrm{Tr}(X) = 0\}$ to $G = SL(2,\mathbb{R}) = \{g \in \mathrm{Mat}_2(\mathbb{R}), \mathrm{Det}(g) = 1\}$ is not surjective.

2. Involutions

Let Φ: $G \to G$ be a Lie group automorphism (an invertible group homomorphism from G to G).

(i) Show that the derivative of Φ is a Lie algebra automorphism.

(ii) Show that the fixed point set is a closed subgroup of G, hence a Lie subgroup. (cf. Cartan's theorem)

(iii) What is the fixed point set of the complex conjugation automorphism of $SU(n)$?

(iv) Find an automorphism of $SU(2n)$ whose fixed point locus is $Sp(n)$.

3. Octonions

The Cayley-Dickson construction is a systematic procedure to obtain a new algebra from an old one, that generalizes the way complex numbers are obtained from the real numbers. Applying it to the complex numbers we get the quaternions, and applying it to the quaternions we get the so-called *octonions*.

A $*$-algebra is an algebra (for us, over $\mathbb{R}$) with a conjugation map $*$ that satisfies $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. We say A is real iff $a^* = a$ for all $a \in A$.

Let A be a $*$-algebra. Define elements of A' to be pairs (a, b) of elements of A, their product by $(a, b)(c, d) = (ac - db^*, a^*d + cb)$, and conjugation by $(a, b)^* = (a^*, -b)$. The following series of facts hold:

- A' is never real
- A is real (hence commutative) $\Leftrightarrow$ A' is commutative
- A is commutative and associative $\Leftrightarrow$ A' is associative

i) Verify the assertions above. Because the quaternions are not commutative, this implies that the octonions are not associative.

Of course, the more familiar definition definition of the quaternions is the $\mathbb{R}$-algebra generated by anti-commuting i, j, k satisfying $i^2 = j^2 = k^2 = ijk = -1$. The products can be recovered from the mnemonic picture $\{i \to j \to k \to\}$. There is a similar mnemonic that gives an operational definition of the octonions, called the Fano plane (Figure 1).

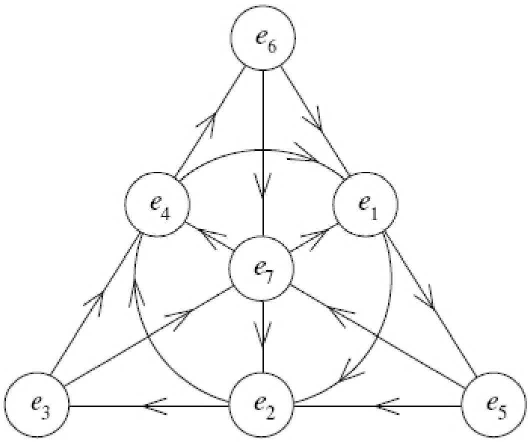

Figure 1: The Fano plane

A useful fact is that the octonions are generated by any triple e_i, e_j, e_k that each squares to -1 and such that e_j anticommutes with e_i, and e_k anticommutes with e_i, e_j and $e_i e_j$. (show this?) We could call this a "basic triple". Such a basic triple is orthonormal w.r.t. the Euclidean inner product on the underlying $\mathbb{R}^8$, which can in fact be expressed invariantly as $\langle a, b \rangle = \mathrm{Re}(a^* b)$.

ii) Like the quaternions, the octonions are equipped with a cross product $a \times b = \frac{1}{2}[a, b]$. Unlike the quaternions, the cross product actually does not make the octonions into a Lie algebra. Why?

iii) It is natural to ask what comes after the octonions. Starting from $\mathbb{R}$ the algebras have been getting progressively worse: $\mathbb{C}$ is no longer real, $\mathbb{H}$ is no longer commutative, and $\mathbb{O}$ is no longer associative. You would be correct in anticipating complete disaster for the next step, called the *sedenions*: show by finding an example that this algebra has zero divisors.

4. Automorphism groups

An automorphism of an algebra is just a map to itself respecting the operations. Such a map is determined by the image of the generators.

Determine the automorphism groups of $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$. They are all groups you are familiar with. Calculate the dimension of the automorphism group of the octonions $\mathbb{O}$. This is not a group you are familiar with (yet). The best you can do for now is give it a name; I propose G_2.