Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 3

Abgabe: Donnerstag, den 5. November 2015,

This week's problem set is based on exercises from *Fulton & Harris*, which are copied here for your convenience.

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1. Exercise 3.32 - The Fourier Transform

o) Prove (or recall the proof) that if the group algebra $\mathbb{C}G$ is identified with the space of functions on G with the function ϕ corresponding to $\sum_{g\in G}\phi(g)e_g$, then the product in $\mathbb{C}G$ corresponds to convolutions of functions:

$$(\phi \star \psi)(g) = \sum_{h\in G} \phi(h)\psi(h^{-1}g)$$

i) Show that $(\widehat{\phi \star \psi})(\rho) = \hat{\phi}(\rho)\hat{\psi}(\rho)$

ii) Prove the *Fourier inversion formula*

$$\phi(g) = \frac{1}{|G|}\sum \dim(V_\rho)\mathrm{Trace}(\rho(g^{-1})\hat{\phi}(\rho))$$

where the sum is over irreducible representations ρ of G.

iii) Prove the *Plancherel formula* for functions ϕ and ψ on G:

$$\sum_{g\in G}\phi(g^{-1})\psi(g) = \frac{1}{|G|}\sum_{\rho} \dim(V_\rho)\mathrm{Trace}(\hat{\phi}(\rho)\hat{\psi}(\rho))$$

2. Exercise 3.38 - The Frobenius-Schur indicator function

Representations can be classified into three types:

- Complex: the character is not real-valued
- Real: the character is real-valued and (there exists a basis in which) the matrices are real
- Quaternionic: the character is real-valued but there exists no basis in which matrices are real[1]

ii) Show that for V irreducible,

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) = \begin{cases} 0 & V \text{ complex} \\ 1 & V \text{ real} \\ -1 & V \text{ quaternionic} \end{cases}$$

[1]Another characterisation, which helps to explain the terminology, is that a representation is real (resp. quaternionic) if and only if it admits a non-degenerate symmetric (resp. skew-symmetric) bilinear form.

3. Exercise 4.4 - The transpose diagram

Set $A = \mathbb{C}\mathcal{S}_d$, so $V_\lambda = Ac_\lambda = Aa_\lambda b_\lambda$.

i) Show that $V_\lambda \cong Ab_\lambda a_\lambda$

ii) Show that V_λ is the image of the map from Aa_λ to Ab_λ given by right multiplication by b_λ. By i) this is isomorphic to the image of $Ab_\lambda \to Aa_\lambda$ given by right multiplication by a_λ.

iii) Show that

$$V_{\lambda'} = V_\lambda \otimes U'$$

where λ' is the conjugate partition to λ and U' is the alternating representation.