Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 2

Abgabe: Donnerstag, den 29. Oktober 2015,

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2. The action(s) of $\mathcal{S}_4$ on the cube

Recall the character table for $\mathcal{S}_4$.

	1	(12)	(123)	(1234)	(12)(34)
Trivial U	1	1	1	1	1
Alternating U'	1	-1	1	-1	1
Standard V	3	1	0	-1	-1
$V \otimes U'$	3	-1	0	1	-1
Another W	2	0	-1	0	2

(i) Verify that the number of elements in each conjugacy class is 1, 6, 8, 6, 3.

By labelling the four long diagonals of a cube with 1,2,3,4, we can realise $\mathcal{S}_4$ as the group of rigid symmetries of the cube. In other words the vector space with one basis element for each long diagonal furnishes the natural permutation representation of $\mathcal{S}_4$ (note that you discovered this in the previous problem set!).

(ii) These rigid symmetries also act on the sets of pairs of opposite faces, faces, vertices, and edges, furnishing representations of dimension 3, 6, 8, and 12. By counting the number of fixed points of an element in a given conjugacy class, compute the characters of these representations (remember/note, in the representation associated to the action of a group on a set, the trace is the number of fixed points). Deduce their decomposition into irreps.

For ease of visualisation:

- A transposition acts as a rotation by 180° around a line through the midpoints of a pair of opposite edges
- A 3-cycle rotates by 120° about a long diagonal
- A 4-cycle rotates by 90° about a line through the midpoints of a pair of opposite faces
- Elements in the class of (12)(34) rotate in the same fashion by 180°.

You can use a physical cube if you need to!

2. Application of the projection formulas: normal modes

Consider a system of three masses attached by springs in a triangle. This system is parametrised by a vector $(x_1, y_1, x_2, y_2, x_3, y_3)$; the x- and y- coordinates of each mass. Let the first mass be at the point $(1, 0)$ and the masses be labelled counterclockwise 1,2,3.

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One central physical concept is the *normal modes* of the system. They are the independent natural patterns in which the system can oscillate (this descriptive definition will suffice for the problem). A normal mode is a triple of vectors in the plane (or a 6-vector) that encodes the direction in which the three masses move.

The key observation is the fact that applying a symmetry transformation to the system also transforms the normal modes. In other words, the normal modes form representations of the symmetry group. This allows us in this case to determine them completely with minimal physical input (in particular, without even knowing Hooke's law!).

In this case the symmetry group is $\mathcal{S}_3 \cong D_3$ the dihedral group of order six. The system is in a six-dimensional representation U_6 that is the tensor product of two representations U_2 and U_3. The first is by 2x2 rotation matrices (possibly combined with a reflection) of the x-y plane (i.e., what happens to the "background" when D_3 acts on the triangle), and the other is by 3x3 permutation matrices which exchange the labels on the masses.

(i) Calculate the character of this representation (by calculating the characters of U_2 and U_3), and determine its irreducible decomposition.

The explicit 6x6 matrices are supplied to you on the last page for your convenience (note that they just look like a rotation by $2\pi/3$ "inserted" in a permutation matrix; this is what tensoring representation matrices does.)

(ii) There are two irreducible one-dimensional representations that appear only once in the decomposition. They must be normal modes. Use the projection formula given in class to project onto the vector generating these one-dimensional representations, and draw what the corresponding motion of the system looks like.

(Tip: if a matrix projects onto a one-dimensional space, it must be of the form vv^T for some (column) vector v ($|\psi\rangle\langle\psi|$ in physics notation). This v is the eigenvector and is not too hard to guess by looking at the matrix.)

(iii) The remaining irreducible representation appears twice in the decomposition. Calculate the matrix that projects onto the two copies of this summand.

To complete the job, we need just a tiny bit of physical intuition: translations in the x- and y-directions will always be normal modes. Those are projected out by the matrices

$$P_X = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

Write down the matrix that projects onto the subspace spanned by the last two normal modes. To see what the motion looks like you can act on any vector of your choice, for example $(1, 0, 0, 0, 0, 0)$. Then a $2\pi/3$ rotation of this motion gives another linearly independent mode.

3. The Mackay quiver of $\mathcal{S}_4$

Last week you classified the finite subgroups of the three dimensional rotation group $SO(3)$. One of them, the octahedral group, corresponding to the symmetries of a cube, is isomorphic to $\mathcal{S}_4$. In this problem we derive the ADE-type Dynkin diagram advertised in the previous assignment for this example.

(i) Recall the double-cover homomorphism $\pi : SU(2) \to SO(3)$ with kernel $\pm I_2$ where I_2 is the 2x2 identity matrix. Argue that the "lift", or pre-image, of $\mathcal{S}_4 \in SO(3)$ (it is called the "binary octahedral" group; denote it $\tilde{\mathcal{S}_4}$) under this homomorphism must contain $-I_2$. Argue similarly that a conjugacy class either lifts to two classes of the same size or one class of twice the size (and say when which one occurs).

(ii) Argue that every irreducible representation of $\mathcal{S}_4$ lifts to an irreducible representation of $\tilde{\mathcal{S}_4}$ and write down the lifted characters.

(iii) There will however be more irreducible representations of $\tilde{\mathcal{S}_4}$, because it has more conjugacy classes. One of them is the fundamental representation coming naturally from the inclusion in $SU(2)$ (let's denote it N)[1]. The following lemma, which you can prove for extra credit, will enable you to find the others.

Lemma: Let V_1, V_2, V_3 be irreducible representations of a finite group G. Then V_3 appears in the decomposition of $V_1 \otimes V_2$ if and only if V_2 appears in $V_1^* \otimes V_3$.

This enables you to use the following strategy [2]:

Start with any irreducible representation, say the trivial U. Then by virtue of the fact that $U \otimes N \cong N$, U must appear in the decomposition of $N \otimes N$, say $N \otimes N \cong U \oplus X$. Calculate the character of X to determine whether it is irreducible, and if it is, repeat the process by looking at $X \otimes N$. If it is not irreducible, pick another irreducible (say U') and start over until you have the right number of them.

Before you get started take note of the definition of the *Mackay quiver* of a subgroup G of $SU(2)$ (a quiver is a directed graph). For every irreducible representation of G, write down a node. If V_2 appears k times in $V_1 \otimes N$, draw k arrow from the node of V_1 to the node of V_2. The algorithm suggested above lets you build the quiver as you go along with your search.

[1]Hint: to compute its character, note that while the double-cover map $\pi : SU(2) \to SO(3)$ is in general somewhat ugly, it takes an acceptable form on simple matrices, for example we can pick axes so that $\pi(\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$. Then you can use the fact that every element of G is conjugate *in* G to a rotation about your favorite axis - this is true because of a fact you proved in the previous assignment.

[2]note that all the representations involved here are equivalent to their dual so you can dispense with the * in the lemma.

Appendix: Matrices for the representation U_6

$$1 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (23) : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$(12) : \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad (123) : \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(132) : \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix} \qquad (13) : \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$