Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 12

Abgabe: Donnerstag, den 28. Januar 2016

1. Unitarity vs. Compactness

Here's a description of "Weyl's unitarity trick" from Fulton&Harris, p. 130:

> The key fact is that *if $\mathfrak{g}$ is any complex semisimple Lie algebra, there exists a (unique) real Lie algebra $\mathfrak{g}_0$ with complexification $\mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}$ such that the simply connected form of the Lie algebra $\mathfrak{g}_0$ is a compact Lie group G.* Thus, restricting a given representation of $\mathfrak{g}$ to $\mathfrak{g}_0$, we can exponentiate to obtain a representation of G, for which complete reducibility holds; and we can deduce from this the complete reducibility of the original representation. For example, while it is certainly not true that any representation ρ of the Lie group $SL(n, \mathbb{R})$ on a (complex) vector space V admits an invariant Hermitian metric (in fact, it cannot, unless it is the trivial representation), we can
> (i) Let ρ' be the corresponding (complex) representation of the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$;
> (ii) by linearity extend the representation ρ' of $\mathfrak{sl}(n, \mathbb{R})$ to a representation ρ'' of $\mathfrak{sl}(n, \mathbb{C})$;
> (iii) restrict to a representation ρ''' of the subalgebra $\mathfrak{su}(\mathfrak{n}) \subset \mathfrak{sl}(n, \mathbb{C})$;
> (iv) exponentiate to obtain a representation ρ'''' of the unitary group $SU(n)$.
> We can now argue that
> If a subspace $W \subset V$ is invariant under the action of $SL(n, \mathbb{R})$,
> it must be invariant under $\mathfrak{sl}(n, \mathbb{R})$; and since $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{R}) \otimes \mathbb{C}$, it follows that
> it will be invariant under $\mathfrak{sl}(n, \mathbb{C})$; so of course
> it will be invariant under $\mathfrak{su}(n)$; and hence
> it will be unvariant under $SU(n)$.
> Now, since $SU(n)$ is compact, there will exist a complementary subspace W' preserved by $SU(n)$; we argue that
> W' will then be invariant under $\mathfrak{su}(n)$; and since $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \otimes \mathbb{C}$, it follows that
> it will be invariant under $\mathfrak{sl}(n, \mathbb{C})$. Restricting, we see that
> it will be invariant under $\mathfrak{sl}(n, \mathbb{R})$, and exponentiating
> it will be invariant under $SL(n, \mathbb{R})$.

To be sure: The existence of the complementary subspace W' follows from the existence of an invariant Hermitian metric on V, obtained by suitably averaging over $SU(n)$ like we did for finite groups. Integration depends on the fact that $SU(n)$ is simply connected and $SL(n, \mathbb{R})$ is connected.

Your task is to prove the statement in parentheses: *A finite-dimensional unitary representation of $SL(n, \mathbb{R})$ is necessarily trivial.*
(i) Let $\mathfrak{k} := \mathfrak{su}(n) \cap \mathfrak{sl}(n, \mathbb{R}) \subset \mathfrak{sl}(n, \mathbb{C})$. Identify $\mathfrak{k}$ as a Lie algebra.
(ii) Identify a subspace $\mathfrak{p} \subset \mathfrak{sl}(n, \mathbb{C})$ such that $\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, while $\mathfrak{su}(n) = \mathfrak{k} \oplus i\mathfrak{p}$ (as vector spaces, not as Lie algebras!)
(iii) By utilizing the two Hermitian metrics, show that for $y \in \mathfrak{p}$, all eigenvalues of $\rho''(y)$ have to be 0.
(iv) Conclude that $\rho = \mathrm{id}_V$.

Please turn over

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2. Root space decomposition of $\mathfrak{so}(5,\mathbb{C})$

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}(5,\mathbb{C}) = \{x \in \mathrm{Mat}(5,\mathbb{C}), x = x^T, \mathrm{tr}x = 0\}$. Let $\mathfrak{h}$ be the abelian subalgebra

$$\mathfrak{h} := \left\{ h = \begin{pmatrix} 0 & ih_1 & 0 & 0 & 0 \\ -ih_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ih_2 & 0 \\ 0 & 0 & -ih_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, h_1, h_2 \in \mathbb{C} \right\} \tag{1}$$

(i) Diagonalize $\mathrm{ad}_{\mathfrak{h}}$ on $\mathfrak{g}$. (You may allow yourself a finite number of lookups. It might also be helpful to revisit $\mathfrak{so}(3,\mathbb{C})$ beforehand.) Express the roots in terms of $\mathfrak{h}^* \ni \lambda_j : h \to h_j$ for $j = 1, 2$. You should find that the roots all lie in $\mathfrak{h}_0^* := \mathrm{span}_{\mathbb{R}}\{\lambda_1, \lambda_2\}$.
(ii) Define an ordering on $\mathfrak{h}_0^*$ by $c^1\lambda_1 + c^2\lambda_2 > 0 \Leftrightarrow$ (the first non-zero $c^j > 0$). Find the highest root.
(iii) Prove that $\mathfrak{g}$ is simple.

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