Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 11

Abgabe: Donnerstag, den 21. Januar 2016

1. Complexification

(i) Let $\mathfrak{g}$ be a real Lie algebra, and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Show that if $\mathfrak{g}^{\mathbb{C}}$ is simple (as a complex Lie algebra), then $\mathfrak{g}$ is simple (as a real Lie algebra).
(ii) Now let $\mathfrak{g}$ be a complex Lie algebra. Assuming that $\mathfrak{g}$ is simple (as a complex Lie algebra), decide whether the underlying real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is necessarily simple. (Proof or counterexample)
(iii) Show that if $\mathfrak{g}$ is any complex Lie algebra, then

$$\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus \mathfrak{g} \tag{1}$$

as a direct product of complex Lie algebras.
Hint: Under restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$, multiplication by the imaginary unit i becomes an endomorphism with some interesting properties.
The standard example for physicists to keep in mind is (cf. Problem 4 on Blatt 6): $\mathfrak{so}(3,1,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2)^{\mathbb{C}}$ is simple, $\mathfrak{so}(3,1,\mathbb{R})^{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. On the other hand, $\mathfrak{so}(4,\mathbb{R}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ already before complexification and is not simple.

2. Reductiveness

Write up the complete arguments for the following statements made in class:
(i) A Lie algebra $\mathfrak{g}$ is reductive iff it admits a non-degenerate ad-invariant symmetric bilinear form.
(ii) This form is unique up to scale iff $\mathfrak{g}$ is either simple or one-dimensional.

3. Two-dimensional Lie Groups

Let G be the group of 2×2 matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \qquad x > 0, y \in \mathbb{R} \tag{2}$$

(i) Verify that G is a Lie group.

(ii) Determine the left- and right-invariant volume forms on G. (The fact that they don't agree is a signal that G is not semi-simple.)
Hint: You can use formulas provided in class, or "guess".

(iii) Determine the Lie algebra associated with G, and calculate the Killing form, with respect to a basis of your choosing.

(iv) Is it true that any compact real two-dimensional Lie group is abelian? Why?

4. Killing- and Volume Form on $SU(2)$

By perusing results and formulas from Übungsblatt 5,
(i) show that the Killing-Form on $\mathfrak{su}(2)$ agrees (up to scale) with the metric induced from the standard metric on $\mathbb{R}^4$ by the identification $SU(2) \cong S^3 \subset \mathbb{R}^4$, and
(ii) calculate (a coordinate expression for) the invariant volume form on $SU(2)$.

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