Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 10

Abgabe: Donnerstag, den 14. Januar 2016

Let F be a field of characteristic 0 and let $\mathfrak{g}$ denote a Lie algebra. Prove all statements.

1. Inner and outer derivations of nilpotent Lie algebras

Let $\mathfrak{g}$ be non-trivial nilpotent Lie algebra of finite dimension d.

(a) $\mathfrak{g}$ contains an ideal $\mathfrak{l}$ of codimension 1.

(b) There exists a positive integer $n \in \mathbb{N}$ such that $C_{\mathfrak{g}}(\mathfrak{l}) \subset \mathfrak{g}_n$, $C_{\mathfrak{g}}(\mathfrak{l}) \not\subset \mathfrak{g}_{n+1}$ and $C_{\mathfrak{g}}(\mathfrak{l}) \setminus \mathfrak{g}_{n+1}$ is not empty, where

$$C_{\mathfrak{g}}(\mathfrak{l}) := \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{l}\}.$$

(c) Conclude that the adjoint representation

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$

is not surjective.
Hint: Write $\mathfrak{g} = \mathfrak{l} \oplus Fx$ for some suitable $x \in \mathfrak{g}$.

Remark: A derivation is called *inner derivation* if it is in the image of ad and *outer derivation* otherwise. In particular, we have seen that a non-trivial nilpotent Lie algebra always admits an outer derivation.

2. Inner derivations of semisimple Lie algebras

Let $\mathfrak{g}$ be semisimple Lie algebra. Then the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is surjective. To prove that, you may pursue the following steps:

(a) $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}(\mathfrak{g})$. Note that this is a general statement which holds for arbitrary Lie algebras.

(b) The Killing form of $\mathrm{ad}(\mathfrak{g})$ agrees with the Killing form of $\mathrm{Der}(\mathfrak{g})$ restricted to the subspace $\mathrm{ad}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$.

(c) Write $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus \mathrm{ad}(\mathfrak{g})^{\perp}$, where $\mathrm{ad}(\mathfrak{g})^{\perp}$ is the orthogonal complement of $\mathrm{ad}(\mathfrak{g})$ with respect to the Killing form. Then $\mathrm{ad}(\mathfrak{g})^{\perp}$ equals zero.

3. Necessary criterion for nilpotency

Let $\mathfrak{g}$ be nilpotent. The Killing form $B\colon \mathfrak{g} \times \mathfrak{g} \to F$ of $\mathfrak{g}$ then is identically zero.

4. Characterization of solvability via the Killing form

$\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ lies in the radical of the Killing form, where the radical S of a symmetric bilinear form $\beta\colon \mathfrak{g} \times \mathfrak{g} \to F$ is given by

$$S := \{x \in \mathfrak{g} \mid \beta(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}.$$

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