Wintersemester 2015/16 — Lie-Gruppen und Darstellungstheorie

Übungsblatt 6

Abgabe: Donnerstag, den 26. November 2015,

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1. Compact symplectic group

Consider $GL_n\mathbb{H}$ the group of quaternionic-linear automorphisms of an n-dimensional quaternionic (i.e. over the ring of quaternions) vector space V. We take V to be in fact a *right* $\mathbb{H}$-module, so that scalars commute with linear transformations. A Hermitian form on V is an $\mathbb{R}$-bilinear form K such that $K(v\lambda, w\mu) = \bar{\lambda}K(v,w)\mu$ where the conjugate of a quaternion $\lambda = a + bi + cj + dk$ is defined as $\bar{\lambda} = a - bi - cj - dk$, and conjugation satisfies $\overline{\lambda\mu} = \bar{\mu}\bar{\lambda}$. The subgroup of $GL_n\mathbb{H}$ preserving the standard Hermitian form $\sum \bar{v}_i w_i$ is called the compact symplectic group $Sp(n)$.

In class we showed that $U(n) = O(2n) \cap Sp_{2n}\mathbb{R}$. Here we deduce the analogous statement for $Sp(n)$:

(*Fulton & Harris*, exercise 7.4)

Regarding V as a complex vector space, show that every quaternionic Hermitian form K has the form

$$K(v,w) = H(v,w) + jQ(v,w)$$

where H is a complex Hermitian form and Q is a skew-symmetric complex linear form on V, with H and Q related by $Q(v,w) = H(vj,w)$, and H satisfying the condition $H(vj,wj) = \overline{H(v,w)}$. Conversely, any such Hermitian H is the complex part of a unique K. If K is standard, so is H, and Q is given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Deduce that

$$Sp(n) = U(2n) \cap Sp_{2n}\mathbb{C}$$

This shows that the two notions of “symplectic” are compatible.

2. Neighborhood of the identity

The following is a preliminary indication of the principle that a Lie group contains most of its information in its tangent space to the identity, its Lie algebra.

(*Fulton & Harris*, exercise 8.1)

Let G be a connected Lie Group, and $U \subset G$ any neighbourhood of the identity. Show that U generates G.

3. The Witt and Virasoro algebra

Consider the vector fields $L_m = -z^{m+1}\partial_z$ on the punctured complex plane.

i) Calculate the Lie brackets

$$[L_n, L_m] = (n-m)L_{n+m}$$

ii) Only a subset of the above vector fields extend to well-defined vector fields on the Riemann sphere $\mathbb{C} \cup \infty$. Determine what they are and show that they form a Lie subalgebra. This Lie algebra is in fact a very familiar one that you should be able to name.

iii) This Lie subalgebra in fact generates the Möbius transformations of the Riemann sphere given by

$$z \mapsto \frac{az+b}{cz+d}$$

Show this by computing $e^{\epsilon L_i} z$ for every L_i in the subalgebra.

iv) Remarkably, you can experience these Möbius transformations in real life. Show that the space of null rays (i.e. null vectors up to rescalings) in Minkowski space is a Riemann sphere, by considering the map μ sending the point $z = u + iv$ to the equivalence class of the null vector

$$(u^2 + v^2 + 1, 2u, 2v, u^2 + v^2 - 1) = (x^0, x^1, x^2, x^3)$$

Compute the action on this equivalence class of a rotation around the x^3 axis and of a boost in the x^3 direction. Determine the corresponding Möbius transformations by rescaling the resulting representative such that you can apply μ^{-1}.

Conclude in particular that the following depiction taken from *Star Wars* of the appearance of the night sky to an accelerating observer, suggesting that the distant stars appear to be spreading away from the direction of motion, is wrong:

v) The Virasoro algebra V is a *central extension* of the Witt algebra U. This means that as a vector space, $V = U \oplus c\mathbb{C}$, $[L_n, c] = 0$ for all n, while the bracket is replaced with

$$[L_n, L_m] = (n-m)L_{n+m} + f(n,m)c.$$

Show that the Jacobi identity implies, up to a change of basis, that

$$f(n,m) = \begin{cases} 0 & \text{if } n+m \neq 0 \\ \gamma(n^3 - n) & \text{if } n = -m \end{cases}$$

for some constant γ.

4. Lorentz group

The Lorentz group $L = O(1,3)$ is the group of linear transformations of $\mathbb{R}^4$ preserving the quadratic form

$$\mathbb{R}^4 \ni (x_0, x_1, x_2, x_3) \mapsto x_0^2 - x_1^2 - x_2^2 - x_3^2 \in \mathbb{R}$$

(i) By considering sign of determinant and x_0, show that L has four connected components.

(ii) By identifying $\mathbb{R}^4$ with $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ via

$$(x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix}$$

and considerations similar to those in $SU(2) \to SO(3)$, construct a homomorphism $SL(2, \mathbb{C}) \to L$. Describe its image and kernel.

This homomorphism of course underlies the phenomenon you studied above about the night sky.

Bonus: Taking inspiration from your previous assignment, find also a homomorphism $SU(2) \times SU(2) \to O(4)$, its image and kernel.

A. Appendix

In case you are not familiar with special relativity, a *boost* with "rapidity" β along the x^3 direction is given by the following matrix

$$\begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix}$$

It represents the effect of a change in velocity of an observer in the x^3 direction.