Lemma 2 Falls $\lambda > \mu$, dann gilt $\forall x \in \mathbb{C}S_n$:

$$p_\lambda x q_\mu = 0 = q_\mu \cdot x \cdot p_\lambda$$

Insbesondere gilt $c_\lambda c_\mu = 0$

(Es gilt auch $c_\mu \cdot c_\lambda = 0$ ~~[illegible]~~, also das brauchen wir nicht.)

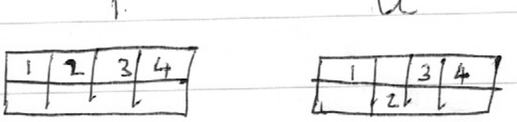
Bew: Wir zeigen $p_\lambda \sigma q_\mu = 0 \quad \forall \sigma \in S_n$. Sei dazu T wie gehabt das ~~Standard~~ kanonische Tableau der Form λ und U das ~~Standard~~ kanonische Tableau der Form μ. Dann ist $\sigma U = U'$ ein weiteres Tableau der Form μ und $\sigma q_\mu \sigma^{-1} = q_{\mu,\sigma}$. Es genügt also zu zeigen, dass $p_\lambda q_{\mu,\sigma} = 0$ für jedes Tableau $\sigma U = U'$ der Form μ.

Taubenschlagprinzip 2 Falls $\lambda > \mu$, so existieren Elemente $a, b \in N$ die in der gleichen Zeile von T und in der gleichen Spalte von U' vorkommen (*)

Die Transposition τ von a und b erfüllt dann

$$p_\lambda q_{\mu,\sigma} = \tau p_\lambda q_{\mu,\sigma} = p_\lambda \tau q_{\mu,\sigma} = -p_\lambda q_{\mu,\sigma} = 0$$

(*) T U'



Herausstreichen der langen Zeilen von T aus U' geht solange gut, wie $\lambda_i = \mu_i$. $\lambda_i > \mu_i$ erzwingt dann doppelt besetzte Spalten in U'.

We are now ready to prove the main theorem.

- First, we claim that $c_\lambda \cdot c_\lambda = N c_\lambda$.

Indeed, this follows from the fact that $\forall \pi \in P, \xi \in Q$

$$\pi c_\lambda c_\lambda \xi = \operatorname{sgn}(\xi) c_\lambda c_\lambda$$

so Lemma 1 implies $\exists N$ s.t. $c_\lambda^2 = N c_\lambda$.

To show that $N \neq 0$, let's calculate it. Note that the trace of right multiplication by c_λ on $\mathbb{C}S_n$ is $n!$ (since $c_\lambda(e) = 1$).
On the other hand c_λ maps any $x c_\lambda \in \mathbb{C}S_n c_\lambda$ to N times itself and anything in the complement into $\mathbb{C}S_n c_\lambda$. So the trace on $\mathbb{C}S_n = \mathbb{C}S_n c_\lambda +$ complement is $\dim V_\lambda \cdot N$. So

$$N = \frac{n!}{\dim V_\lambda}.$$

- To show that V_λ are irreducible, assume $W \subset V_\lambda$ is invariant subspace, and consider $c_\lambda W$. Since $c_\lambda V_\lambda = \mathbb{C} c_\lambda$ by Lemma 1, we must have either $c_\lambda W = \mathbb{C} c_\lambda$ or $c_\lambda W = 0$.

If $c_\lambda W = \mathbb{C}c_\lambda$, $\exists\ w \in W$ s.t. $c_\lambda w = c_\lambda$.

But since any $v \in V_\lambda$ is of the form $v = xc_\lambda$ we learn

$$v = xc_\lambda = xc_\lambda w \in W.$$

So $V_\lambda = W$.

If $c_\lambda W = 0$, then $c_\lambda W^\perp = \mathbb{C}c_\lambda$, where $W^\perp$ is complementary invariant subspace. But then by the argument we just gave $W^\perp = V$, so $W = 0$.

· Finally, we show that if $\lambda > \mu$, then V_λ and V_μ are not isomorphic. Indeed, $c_\lambda V_\lambda = \mathbb{C}c_\lambda$ but $c_\lambda V_\mu = p_\lambda(q_\lambda \mathbb{C}S_n p_\mu)q_\mu = 0$ by Lemma 2.

So $V_\lambda \neq V_\mu$.

Thus, we have found as many irreps as there are conjugacy classes. Since there can't be more, we must have found all of them. The proof is complete.

Beispiel: Die 2-dimensionale Darstellung von S_3 gehört zur Partition $3 = 2+1$, i.e. das Young-Diagramm ist $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$:

Das kanonische Tableau ist $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, der zugehörige Young-Symmetrisierer ist $c_\lambda = (1+(12))(1-(13)) = 1+(12)-(13)-(132)$.

Hierbei bezeichnen runde Klammern Zykel $(1\,\sigma(1)\,\sigma^2(1)\dots)$, deren Verknüpfung man z.B. durch die Wirkung auf $N = \{1,2,3\}$ bestimmt:

$$(12)(13)\{1,2,3\} = (12)\{3,2,1\} = \{3,1,2\} = (132)\{1,2,3\}$$

Benutzen wir für $\mathbb{C}S_n$ die geschweifte Notation, so ist

$$Y_\lambda(\{1,2,3\}) = \{1,2,3\} + \{2,1,3\} - \{3,2,1\} - \{3,1,2\} =: e_1$$

$$Y_\lambda(\{1,3,2\}) = (23)(Y_\lambda\{1,2,3\}) = \{1,3,2\} + \{3,1,2\} - \{2,3,1\} - \{2,1,3\} =: e_2$$

$$(12)\, e_1 = \{2,1,3\} + \{1,2,3\} - \{3,1,2\} - \{3,2,1\} = e_1$$

$$(12)\, e_2 = \{2,3,1\} + \{3,2,1\} - \{1,3,2\} - \{1,2,3\} = -e_2 - e_1$$

$$(23)\, e_1 = e_2$$

$$(23)\, e_2 = e_1$$

~~Da (12), (23) von S_3 erzeugen, folgt bereits, dass e_1 und e_2 $V_{\square}$ aufspannen und aus $tr_{\square}(23) = tr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$ können wir die folgt die Identifikation~~

Da (12) und (23) S_3 erzeugen, folgt bereits, dass e_1 und e_2 $V_{(2,1)}$ aufspannen. Aus $\operatorname{tr}(23) = \operatorname{tr}\begin{pmatrix}0&1\\1&0\end{pmatrix} = 0$ und

$\operatorname{tr}(23)(12) = \operatorname{tr}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&-1\\0&-1\end{pmatrix} = \operatorname{tr}\begin{pmatrix}0&-1\\1&-1\end{pmatrix} = -1$ folgt die Identifikation mit der 2-dimensionalen Darstellung aus Kapitel 1. Die mit Young-Diagrammen verzierte Charaktertafel ist also jetzt:

	$(1,1,1)$	$(2,1)$	(3)
(3)	1	1	1
$(2,1)$	2	0	-1
$(1,1,1)$	1	-1	1

Hätten wir statt dem kanonischen das andere Standardtableau $(23)\begin{smallmatrix}1&2\\3&\end{smallmatrix} = \begin{smallmatrix}1&3\\2&\end{smallmatrix}$ benutzt, so wäre der entsprechende Young-Symmetrisierer

$$c'_{\lambda} = (1+(13))(1-(12)) = 1+(13)-(12)-(123)$$

und

$$e'_1 = \{1,2,3\} + \{3,2,1\} - \{2,1,3\} - \{2,3,1\}$$

$$e'_2 = \{1,3,2\} + \{2,3,1\} - \{3,1,2\} - \{3,2,1\} = (23)\, e'_1$$

$$(12)\, e'_1 = -e'_1 - e'_2$$

$$(12)\, e'_2 = e'_2$$

	$\{1,2,3\}$	$\{1,3,2\}$	$\{2,1,3\}$	$\{2,3,1\}$	$\{3,1,2\}$	$\{3,2,1\}$
e_1	1	0	1	0	-1	-1
e_2	0	1	-1	-1	1	0
e_1'	1	0	-1	-1	0	1
e_2'	0	1	0	1	-1	-1
$\square\square\square$	1	1	1	1	1	1
$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$	1	-1	-1	1	1	-1

Check: e_1, e_2 and e_1', e_2' are linearly independent.

(*) Der Beweis der Aussagen über eine Basis von V_λ und $\mathrm{Hom}(V_\lambda, \mathbb{C}S_n) = V_\lambda^*$ an dieser Stelle stellt sich als zu aufwendig heraus.

Die Idee ~~ist~~ wäre gewesen, zunächst die lineare Unabhängigkeit der Standardtableaus zu zeigen (was einfacher ist als die Erzeugung) und anschliessend kombinatorisch zu verifizieren, dass

$$n! = |S_n| = \sum_{\substack{\lambda \text{ Partition} \\ \text{von } n}} \left(\#\left\{\begin{matrix}\text{Standardtableaus}\\ \text{der Form } n\end{matrix}\right\}\right)^2 .$$

Siehe z.B. B. Simon, Reps of finite & compact groups

G.D. James, The representation theory of the symmetric groups.

(aus letzterem Buch stammt wohl auch der folgende Algorithmus für die Charaktertafel, der ohne symmetrische Funktionen auskommt.)

§3.3. Die Charaktertafel der symmetrischen Gruppe

~~Lie groups and representation theory – Lecture 5~~

~~Representation theory of the symmetric group II – Characters~~

In the previous lecture, we reviewed the fact that conjugacy classes $C_{\vec{k}}$ in the symmetric group S_n are uniquely determined by the cycle type

$$\vec{k} = (k_1, k_2, \dots) \qquad \sum i k_i = n$$

where k_i := # cycles of length i.

We then showed how to associate irreducible representations V_λ to "Young tableaus of shape λ", where,

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \qquad \sum \lambda_j = n$$

is a partition of n.

$$V_\lambda = \mathbb{C}S_n c_\lambda = \mathbb{C}S_n p_\lambda q_\lambda .$$

where $\displaystyle p_\lambda = \sum_P \pi \qquad q_\lambda = \sum_Q \operatorname{sgn}(\xi)\, \xi$

and P, Q are row, column stabilizer of given Young tableau.

We now wish to calculate the irreducible characters

$$\chi_\lambda(\vec{k}) = \operatorname{tr}_{V_\lambda} \sigma \quad , \sigma \in C_{\vec{k}} .$$

not quite the same definition, but OK

As notation and terminology, V_λ is known as "Specht module" (module being synonymous with representation in this course).
The row stabilizer $P = P_{\hat{\lambda}}$, whose embedding in S_n depends on tableau $\hat{T}$ is known as Young subgroup.

- We record that $|P| = \prod_j \lambda_j!$, and that the size of conjugacy class is given by the formula

$$|C_{\vec{k}}| = \frac{n!}{\text{size of stabilizer/centralizer}} = \frac{n!}{\prod_i i^{k_i} k_i!}$$

- We know from general theory that ~~also~~ irreducible characters are orthogonal w.r.t. particular inner product

$$\sum_{\vec{k}} \bar{\chi}_\lambda(\vec{k}) \frac{|C_{\vec{k}}|}{n!} \chi_\mu(\vec{k}) = \delta_{\lambda\mu}$$

~~(the formulas imply that χ's are all real. I'll omit the $^-$)~~

- We ~~write this as matrix formula. We have already~~ ordered partitions (lexicographically)

$$\lambda > \mu \quad \text{if} \quad \lambda_i - \mu_i > 0 \quad \text{for first non-zero.}$$

Identifying cycle types with partitions allows us to also order $\vec{k}$'s and even compare λ's with $\vec{k}$'s.

With this ordering, we let $X = (X_{\lambda\vec{k}})$ be the character table as a matrix in which rows are indexed by irreps (λ) and columns by conjugacy classes ($\vec{k}$).

Then, let Σ be diagonal matrix

$$\Sigma_{\vec{k}\vec{l}} = \delta_{\vec{k}\vec{l}} \frac{|C_{\vec{k}}|}{n!}$$

so that orthogonality of characters becomes

$$X \Sigma X^T = \mathbb{1}.$$

—

The main step towards calculating X is to introduce the following "auxiliary representations W_λ ~~which~~ also indexed by partitions, but generally reducible. With same notation as before,

$$W_\lambda = \mathbb{C} S_n p_\lambda.$$ i.e. only symmetrize the rows.

Calculation of character of W_λ is fairly straightforward. Let $\sigma \in C_{\vec{k}}$. Then

$$\phi_\lambda(\vec{k}) = \operatorname{tr}_{W_\lambda} \sigma = \#\left\{ \tau \in S_n,\ \sigma\tau = \tau\pi, \text{ some } \pi \in P_\lambda \right\} \Big/ \tau P_\lambda$$

(in words, we want the number of tableaux such that acting with σ can be repaired by a row permutation, but we identify two such if they are related by a row permutation) this is,

$$\mathrm{tr}_{W_\lambda}\sigma = \#\left\{\tau\in S_n \; , \; \tau^{-1}\sigma\tau\in P_\lambda\right\} / |P_\lambda|$$

$$= \frac{|P_\lambda\cap C_{\vec{k}}|}{|P_\lambda|}\cdot|\text{stabilizer of }\sigma|$$

$$\psi_\lambda(\vec{k}) = \frac{|P_\lambda\cap C_{\vec{k}}|\;\; n!}{|P_\lambda|\,|C_{\vec{k}}|}$$

The W_λ is an example of a so-called induced representation (here: the rep of S_n induced by the ~~subgroup~~ ~~P_λ~~ trivial rep of the subgroup P_λ and the formula for $\psi_\lambda(\vec{k})$ is an example of Frobenius reciprocity. Siehe Kapitel 2.
~~[See end of lecture 3 which we did not cover in class.~~

I'll give a more explicit formula for $\psi_\lambda(\vec{k})$ in terms of symmetric functions, below.
In fact, ~~I~~ most of the literature also does the next step, the decomposition of $\psi_\lambda(\vec{k})$ into irreducibles, in terms of symmetric functions.
However, this is unnecessary.
A noteworthy thing however is that $\psi_\lambda(\vec{k})=0$ if $\vec{k}>\lambda$. (in the conventions of before maximal length of cycle in S_{λ_i} is λ_i.)

Therefore, the matrix (this doesn't play any role actually...)

$$\Psi = (\psi_\lambda(\vec{k}))$$

is in fact lower triangular.

Now on general grounds we know that

$$W_\lambda \cong \bigoplus V_\mu^{\oplus K_\lambda^\mu} \qquad (*)$$

for some non-negative integer degeneracies $K_\lambda^\mu \in \mathbb{N}_{\geq 0}$ known as Kostka numbers.

Claim: The matrix $K = (K_\lambda^\mu)$ is upper triangular and invertible (i.e. $K_\lambda^\mu = 0$ if $\mu < \lambda$ and $K_\lambda^\lambda > 0$.)

The second claim follows immediately from the fact that

$$V_\lambda = W_\lambda q_\lambda.$$

i.e. right-multiplication by q_λ defines a non-zero map in $\mathrm{Hom}_{\mathbb{C}S_n}(W_\lambda, V_\lambda)$

To check the first claim, we verify that if $\lambda > \mu$

$$\mathrm{Hom}_{\mathbb{C}S_n}(W_\lambda, V_\mu) = \mathrm{Hom}_{\mathbb{C}S_n}(\mathbb{C}S_n p_\lambda, \mathbb{C}S_n p_\mu q_\mu) = 0$$

Indeed, to specify such a map, say $f: W_\lambda \to V_\mu$, it's enough to give $f(p_\lambda) = x\, p_\mu q_\mu$, $x \in \mathbb{C}S_n$

But since $p_\lambda^2 \propto p_\lambda$ we find

$$f(p_\lambda) = \frac{1}{|P_\lambda|} f(p_\lambda^2) = \frac{1}{|P_\lambda|} p_\lambda f(p_\lambda)$$

$$= \frac{1}{|P_\lambda|} p_\lambda x\, p_\mu q_\mu = 0 \quad \text{if } \lambda > \mu \text{ by Lemma 2}$$

As a consequence of $(*)$,

$$\psi_\lambda(\vec{k}) = \sum K_\lambda^{\ \mu} \chi_\mu(\vec{k})$$

i.e.

$$\Psi = K X$$

Therefore,

$$\Psi \Sigma \Psi^T = K X \Sigma X^T K^T \quad \text{(inner product of characters computed in terms of irreducibles)}$$

$$\Psi \Sigma \Psi^T = K K^T.$$

It's not hard to check that an upper triangular matrix K with positive diagonal entries is uniquely determined by this equation.
Integrality follows from general principles.

At the end,

$$X = K^{-1} \Psi$$

and we are done.

Addendum: Symmetric functions

- First useful observation is the generating function for the number of partitions:

$$\sum_{n=0}^{\infty} N(n) q^n = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} q^{nk}$$

$$= \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

– closely related to Dedekind η-function.

$$N(n) \sim \frac{1}{4\sqrt{3}\, n} e^{\pi\sqrt{2/3}\, n}$$

- Second, let's calculate $|P_\lambda \cap C_{\vec{k}}|$. We consider conjugacy classes in

$$P_\lambda = \prod_{j=1}^{s} S_{\lambda_j}$$

they are labelled by collections (r_{ij}) s.t.

$$\lambda_j = \sum_i i\, r_{ij}$$

To get an element in conjugacy class $C_{\vec{k}}$ of S_n, we need

$$\sum_j r_{ij} = k_i.$$

Then

$$|P_\lambda \cap C_{\vec{k}}| = \sum_{(r)} \prod_j \frac{\lambda_j!}{\prod_i i^{r_{ij}} r_{ij}!}$$

So

$$\psi_\lambda(\vec{k}) = \frac{1}{\prod \lambda_j!} \prod i^{k_i} k_i! \sum_{(r)} \prod_j \frac{\lambda_j!}{\prod_i i^{r_{ij}} r_{ij}!}$$

$$= \sum_{(r)} \prod_i \frac{k_i!}{\prod_j r_{ij}!} \qquad \sum_j r_{ij} = k_i \qquad \sum_i i r_{ij} = \lambda_j$$

Frobenius observation is that this combination of multinomial coefficients is also the coefficient of

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_s^{\lambda_s}$$

in

$$\underbrace{(x_1 + \cdots + x_s)^{k_1} (x_1^2 + \cdots + x_s^2)^{k_2} \cdots}$$

$$\psi_\lambda(\vec{k}) = \left[P^{(\vec{k})} \right]_\lambda \quad - \text{ coefficient of } x^\lambda.$$

which is the beginning of a long story involving symmetric functions, at the end of which one proves the Frobenius character formula

$$\chi_\lambda(\vec{k}) = \left[\Delta P^{(\vec{k})}\right]_l$$

$$= \left(\text{coefficient of } x^l \text{ in } \Delta \cdot P^{(\vec{k})}\right)$$

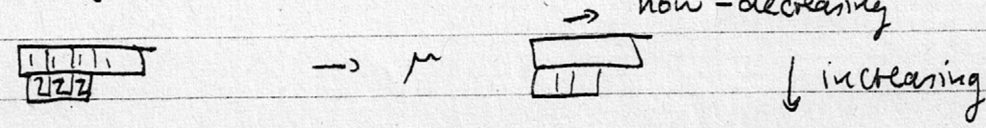
where $P^{(\vec{k})}$ as above,

$$\Delta = \prod_{i<j} (x_i - x_j)$$

(Vandermonde determinant of $\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{s-1} \\ 1 & x_2 & & & \\ & & & & \\ 1 & x_s & \cdots & & x_s^{s-1} \end{pmatrix}$

and $l = (\lambda_1 + s - 1, \lambda_2 + s - 2, \ldots \lambda_{s-1} + 1, \lambda_s)$

The gist of it is to define Kostka numbers combinatorially as ways of filling up Young diagrams

λ: 

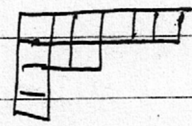
and then show that corresponding decomposition of $\psi_\lambda(\vec{k})$ is irreducible character.

(what I described is obviously faster, though it might give less information.)

That $K_\lambda^\mu = \mathrm{Hom}(V_\mu, W_\lambda)$ follows as corollary.)

Among the consequences of the Frobenius character formula is the so-called Hook lenght formula:

$$\dim V_\lambda = \frac{n!}{\prod_{\text{boxes}} (\text{Hook lengths})}$$



$$\begin{array}{|c|c|c|c|} \hline 7 & 4 & 2 & 1 \\ \hline 4 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \frac{8!}{7\cdot 4\cdot 2\cdot 4\cdot 2} = 6\cdot 5\cdot 3 = 90.$$

(whose equality with number of standard tableaux is hard to prove, combinatorially.)

as well as a variety of other combinatorial results (Murnaghan-Nakayama rule,..etc.)