Kapitel 1 Einführung, Übersicht und Zusammenfassung

§ ~~Abschnitt~~ 1.1. Darstellungstheorie

Def. Eine (lineare) Darstellung einer Gruppe G ist ein Vektorraum V zusammen mit einem Homomorphismus

$$\rho : G \longrightarrow GL(V) \qquad \left(\text{Gruppe der linearen Automorphismen}\right)$$

"abstrakte Gruppe" — multiplikative Verknüpfung

"konkrete Gruppe" — Matrixmultiplikation

$$\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2)$$

Lemma: Sei $e \in G$ das neutrale Element ~~die Einheit~~ in G, so ist $\rho(e) = id_V$ die Identität auf V.

Bew.: $\forall\, v \in V$ gilt

$$\begin{aligned} \rho(e)v &= \rho(e)\,\rho(e)\,\rho(e)^{-1}v \\ &= \rho(e^2)\,\rho(e)^{-1}v \\ &= \rho(e)\,\rho(e)^{-1}v = v. \end{aligned}$$

<u>Bemerkung</u> Ohne weiteres Angaben werden alle Vektorräume über dem Körper $\mathbb{C}$ der komplexen Zahlen angenommen (d.h., Charakteristik 0, metrisch vollständig und algebraisch abgeschlossen sind alle wichtig). Manchmal ist es auch interessant, reelle Vektorräume (über $\mathbb{R}$) zu betrachten.
Weitere Annahmen werden später hinzugefügt.

Als semi-philosophischer Kommentar, die ich mir nur schwer verkneifen kann, sei ~~von~~ noch hinzugefügt, dass vom Standpunkt eines Physikers natürlich G "konkret" ist im Sinne einer operationellen Transformation eines physikalischen Systems, während insbesondere Experimentalphysiker natürlich (V, ρ) als "abstrakte Mathematik" ansehen

~~<u>Remarks:</u> • Without any specification, all our vector spaces will be over $\mathbb{C}$, the field of complex numbers. Sometimes we also find it interesting to consider vector spaces over $\mathbb{R}$.~~
~~• Other assumptions (about G, V, ρ) will be added later.~~

<u>Example:</u> $G = \mathbb{Z}/2\mathbb{Z} = \{e, \sigma\} \quad \sigma \cdot \sigma = e$

$$V = \mathbb{C} \qquad GL(V) = \mathbb{C}^{*} = \mathbb{C} \setminus \{0\}$$

$\rho(e) = id_V = 1$ as we've just seen

$$\rho(\sigma)^2 = \rho(e) = 1$$

$$\rightsquigarrow \quad \rho(\sigma) = \begin{cases} +1 & \rho_+ \\ -1 & \rho_- \end{cases}$$

Verwandte Begriffe

~~Associated notions:~~ Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of a group G

* a map (or G-linear map, or G-morphism) from (V_1, ρ_1) to (V_2, ρ_2) is a map
$$\varphi : V_1 \to V_2$$
of vector spaces such that $\forall g \in G$
$$\varphi \circ \rho_1(g) = \rho_2(g) \circ \varphi$$

* trivial: $\varphi = id : V_1 \circlearrowleft$
$$0 : V_1 \to 0$$

* $\dim V = \dim \rho$.

* (V_1, ρ_1) and (V_2, ρ_2) are equivalent (or isomorphic) representations if $\exists$ a G-isomorphism $\varphi : V_1 \to V_2$.

* (V_1, ρ_1) is a subrepresentation of (V_2, ρ_2) if there exists an injective G-linear map, in other words if V_1 is a subvectorspace of V_2 and $\rho_1 = \rho_2|_{V_1}$.

* The direct sum $V_1 \oplus V_2$ carries a natural representation

$$(\rho_1 \oplus \rho_2)(g) = \rho_1(g) \oplus \rho_2(g)$$

so does the tensor product $V_1 \otimes V_2$

* a representation (V, ρ) is called irreducible if it does not ~~conta~~ contain any non-trivial subrepresentations (i.e. there are no invariant subspaces).

<u>Examples</u>: $G = \mathbb{Z}/2$

- $(\mathbb{C}, \rho_-) \otimes (\mathbb{C}, \rho_-) \cong (\mathbb{C}, \rho_+)$

- $(\mathbb{C}^2, \rho_2) \cong (\mathbb{C}, \rho_-) \oplus (\mathbb{C}, \rho_+)$

with $\rho_2(\bar{1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

* **NB.** By the "trivial representation", one usually means $V = \mathbb{C}$, $\rho(g) = 1 \quad \forall g$.
Having the trivial representation as subrepresentation is non-trivial.

Example. $G = S_3$, symmetric group on 3 elements is represented on $\mathbb{C}^3$ by choosing a basis (b_1, b_2, b_3) and putting

$$\rho(\sigma)\, b_i = b_{\sigma(i)}$$

$$\text{(i.e. } \rho(\sigma)(v)^i = v^{\sigma^{-1}(i)} \qquad \text{when } v = \sum v^i b_i$$

This has invariant subspace $\mathbb{C} \hookrightarrow \mathbb{C}^3$ generated by $b_1 + b_2 + b_3$.

The complementary subspace (orthogonal complement w.r.t standard inner product) is ~~generated~~ spanned by $b_1 - b_2$, $b_2 - b_3$ in which basis one has representation matrices

$$\rho_2(123) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad \rho_2(12) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\cancel{\rho_2(132) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}} \qquad \rho_2(13) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} =$$

$$\rho_2(132) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_2(123)\,(b_1-b_2) = b_2-b_3$$

$$b_2-b_3 = b_3-b_1 = -(b_1-b_2)-(b_2-b_3)$$

Matrix Darstellung: $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

$$\rho_2(12)\,(b_1-b_2) = b_2-b_1$$

$$(b_2-b_3) = b_1-b_3 = b_1-b_2+b_2-b_3$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\rho_2(13)\,(b_1-b_2) = b_3-b_2$$

$$b_2-b_3 = b_2-b_1$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Diese Darstellung ist irreduzierbar (betrachte die Eigenräume von $\rho_2(123)$ unter $\rho_2(12)$) und wir erhalten die Zerlegung

One can check that ρ_2 is irreducible (this follows from the fact that eigenvalues of $\rho_2(123)$ are both non-trivial) and we have the decomposition

$$(\mathbb{C}^3, \rho) = (\mathbb{C}^2, \rho_2) \oplus (\mathbb{C}, \rho_+)$$

trivial representation

of $\mathbb{C}$ the "permutation representation" into irreducible ~~comp~~ subrepresentation

~~The first goal of this course is to establish (or remind ourselves) that a similar~~

NB: There is another rep of S_3, not equivalent to either ρ_+, ρ_2, defined by

$$\rho_-(123) = 1 \quad \rho_-(12) = -1.$$

alternating representation

The first goal of this course is to establish (or remind ourselves of) the similar result for any finite group (i.e. complete reducibility) and to describe how all the irreps can be obtained

I understand that many of you have seen this result already, and those of you who haven't will soon see that it is rather simple. So this will be fairly quick.

I was then planning to look specifically at the representation theory of the symmetric group S_n in general, and describe the irreps of that in terms of Young diagrams explicitly.

I will skip any or both of these topics if voted down by a majority of students, though certainly I can say is worthwhile baggage for any aspiring mathematician.

It is perhaps worthwhile to point out that understanding (and classifying) representations is not the same (although it helps) in understanding and classifying groups. The analogue of irreducible is "simple" (no normal subgroups). For finite groups, this is indeed daunting. (Cf. Monster group, $8 \cdot 10^{53}$ elem

However, we can do more if we assume extra structure, specifically a continuous structure.

~~Abschnitt~~ §1.2. Kontinuierliche Gruppen

So what I was planning to do after rep. theory of finite groups is to repeat the story for compact topological groups. The theory is virtually the same as for finite groups, ~~however~~ and is known as Peter-Weyl theorem. However turns out one needs an analytical result, the existence of invariant measure, Haar measure. Again I won't do it if you object, but I was planning to cover it for my own benefit.

With this done, we can attack the actual subject of the course, Lie groups.

Def: Lie group G is a group that is also a differentiable manifold such that group operations are differentiable.
(I'll discuss more precise notions when we get to it.)
For now, I want to do an example.

Example: $G = SU(2) = \left\{ g \in Mat_2(\mathbb{C}),\ g^{+}g = id \atop \det g = 1 \right\}$

<u>Fact:</u> $SU(2) \cong S^3$ (the three-dimensional sphere)

$$g = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \qquad g^{\dagger} = \begin{pmatrix} \bar z_1 & \bar z_3 \\ \bar z_2 & \bar z_4 \end{pmatrix}$$

$$g^{\dagger} g = id \quad \leadsto \quad |z_1|^2 + |z_1|^2 = |z_3|^2 + |z_4|^2 = 1$$

$$z_1 \bar z_3 + z_2 \bar z_4 = \left(z_3 \bar z_1 + \bar z_2 z_4 = \right) 0$$

$$\det g = 1 \quad \leadsto \quad z_1 z_4 - z_2 z_3 = 1$$

$$|z_1|^2 z_4 + |z_2|^2 z_4 = \bar z_1$$

$$z_4 = \bar z_1.$$
$$z_3 = -\bar z_2$$

$$g = \begin{pmatrix} z_1 & z_2 \\ -\bar z_2 & \bar z_1 \end{pmatrix} \qquad \det g = 1 \qquad |z_1|^2 + |z_2|^2 = 0$$

$$z_1 = x_1 + i x_2 \qquad z_2 = x_3 + i x_4.$$

$$S^3 = \{ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \} \subset \mathbb{C}^2 = \mathbb{R}^4$$

whatever is your notion of knowledge of manifolds, you will accept that S^3 is such a thing (a <u>real</u> compact 3-dim manifold).

Really the only thing we need from dif. manifolds, ~~is~~ ~~the~~ at least for a while is the notion of <u>tangent space</u>. ~~or~~ tangent space at identity

Say $g = id + \varepsilon a$, ε infinitesimal, $\in \mathbb{R}$

$$g^\dagger g = id + \varepsilon(a^\dagger + a) + \mathcal{O}(\varepsilon^2)$$

$$= id \iff a^\dagger + a = 0$$

$$\det g = 1 + \varepsilon \operatorname{tr} a + \mathcal{O}(\varepsilon^2)$$

$$\rightsquigarrow \operatorname{tr} a = 0.$$

indeed, space of 2×2 anti-hermitian traceless matrices is 3-dimensional $\mathbb{R}$ vector space

$$\mathfrak{g} = \left\{ \begin{pmatrix} x_3 & x_1 + i x_2 \\ x_1 - i x_2 & -x_3 \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R} \right\} \cong \mathbb{R}^3.$$

(Lie algebra of G)

there is a lot more to say here; instead I want to conclude the lecture by giving two examples of representations of $SU(2)$.

Wir werden später zumindestens die Darstellungstheorie von $SU(2)$ vollständig behandeln

An dieser Stelle sei nebst der trivialen ($\rho = id$) und der definierenden 2-dim. Darstellung

$$\rho_2 : SU(2) \longrightarrow GL(2, \mathbb{C})$$

die folgende interessante, 3-dimensionale, sog. adjungierte Darstellung auf der Lie Algebra erwähnt:

$$\rho_3 : SU(2) \longrightarrow GL(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$$

$$\rho_3(g)\, a = g a g^{+}$$

(wegen Linearität, $\operatorname{tr}(g a g^{+}) = \operatorname{tr}(g^{+} g a) = \operatorname{tr} a = 0$ ist dies eine Darstellung).

Ausserdem ist ρ_3 eine reelle Darstellung, und da wegen

$$\det a = -x_1^2 - x_2^2 - x_3^2$$

$$\det g a g^{+} = \det a$$

erhält ρ_3 das Standard innere Produkt auf $\mathbb{R}^3 \cong \mathfrak{g}$.

ρ_3 liefert somit einen Homomorphismus

$$SU(2) \longrightarrow SO(3)$$

(genauer gesagt landet ρ_3 zunächst in $O(3)$ aber wegen der Stetigkeit muss das Bild zusammenhängend sein)

Man prüfe nach: • ρ_3 ist surjektiv (jede Hermitesche Matrix ist unitär diagonalisierbar)

aber ρ_3 ist nicht injektiv:

$$\rho_3(-id) = id.$$

$-id$ ist der gesamte Kern und daher $SO(3) = SU(2)/\mathbb{Z}_2$

Geometrisch $SO(3) = S^3/\mathbb{Z}_2 = \mathbb{RP}^3$.

Nach diesem kurzen Überblick kehren wir nun zunächst zu den endlichen Gruppen und deren Darstellungstheorie zurück.

Kapitel 2 Darstellungstheorie endlicher Gruppen

Zunächst noch eine weitere (zusätzlich zu direkter Summe etc.) Operation auf Darstellungen: Falls $\rho: G \to GL(V)$ eine Darstellung einer Gruppe G ist, so trägt der Dualraum $V^* = \mathrm{Hom}(V, \mathbb{C})$ in natürlicher Weise die sogenannte duale Darstellung, welche definiert wird durch

$$\rho^*: G \to GL(V^*)$$

$$\rho^*(g) = \rho(g^{-1})^* \quad \leftarrow \text{duale linear Abbildung}$$

was im Klartext heisst $\lambda \in V^*$ $\lambda(v) \in \mathbb{C}$.

$$\rho^*(g)(\lambda)(v) = \rho(g^{-1})^*(\lambda)(v) = \lambda(\rho(g^{-1})v)$$

(i) Die Paarung $V^* \otimes V \to \mathbb{C}$, $(\lambda, v) \mapsto \lambda(v)$ ist invariant unter $\rho^* \otimes \rho$.

(ii) Der Witz ist, dass wir das Inverse benötigen, um überhaupt eine Darstellung zu bekommen: $*$ und $^{-1}$ sind kontravariant sodaß die Verknüpfung kovariant wird.

$$\rho^*(g_1 g_2) = \rho(g_2^{-1} g_1^{-1})^* = \left(\rho(g_2^{-1})\, \rho(g_1^{-1})\right)^*$$

$$= \rho(g_1^{-1})^* \rho(g_2^{-1})^* = \rho^*(g_1)\, \rho^*(g_2).$$

- Identifiziert man (für endlich-dimensionale Darstellungen) lineare Abbildungen mit Matrizen (nach Wahl einer Basis) so gilt

$$\rho^*(g) = \rho(g)^{-T}$$

- Für <u>unitäre Darstellungen</u> (mehr sogleich) gilt $\rho^*(g) = \overline{\rho(g)}$ (komplexe Konjugation, kovariant)
  Daher identifizieren Physiker für gewöhnlich duale und konjugierte Darstellung. In dieser Vorlesung wollen wir aber etwas vorsichtiger sein und uns über die Abhängigkeit von der Wahl einer invarianten Sesquilinearform Rechenschaft ablegen.

Im folgenden stellen wir uns vor allem endlich-dimensionale Darstellungen vor. Wir begründen diese Einschränkung am Anfang des nächsten Kapitels.

§2.1. Lemma von Schur

Lemma: (Schur's Lemma) Let (V_1, ρ_1) and (V_2, ρ_2) be representations of G, and $\varphi \in \mathrm{Hom}_G(V_1, V_2)$ a G-morphism.

(i) $\mathrm{Ker}\,\varphi$, $\mathrm{Im}\,\varphi$ and $\mathrm{Coker}\,\varphi$ are naturally also representations of G.

(ii) If V_1 & V_2 are irreducible, then φ is invertible or 0.

(iii) If φ' is another such G-morphism, V_1, V_2 irreducible and $\varphi \neq 0$, then $\exists\ \alpha \in \mathbb{C}$ s.t.

$$\varphi' = \alpha \varphi.$$

Pf. (i) trivial.

$$\varphi(v) = 0 \Rightarrow \varphi(\rho_1(g) v) = \rho_2(g)\, \varphi(v) = 0$$

etc.

(ii) If V_1 is irreducible, and $\mathrm{Ker}\,\varphi \neq V_1$, then $\mathrm{Ker}\,\varphi = 0$ so φ is injective. If V_2 is irreducible, $\mathrm{Im}\,\varphi \neq 0$ then $\mathrm{Im}\,\varphi = V_2$, so φ is surjective.

(iii) so $\varphi^{-1}\varphi' : V_1 \to V_1$. Because we working over $\mathbb{C}$, ~~there~~ is this map has an eigenvalue, i.e. $\alpha \in \mathbb{C}$ such that

$$\varphi^{-1}\varphi' - \alpha\, \mathrm{id}_{V_1}$$

is not invertible. But since $\varphi^{-1}\varphi' \in \mathrm{Hom}_G(V_1, V_1)$ it must then be 0 by (ii). So

$$\varphi^{-1}\varphi' = \alpha\, \mathrm{id}_{V_1}.$$

$\square$

Remark: Often, Schur's Lemma is stated as

$$\mathrm{Hom}_G(V_1, V_2) = \mathrm{Hom}(V_1^* \otimes V_2, \mathbb{C})^G = \begin{cases} 0 & V_1 \not\cong V_2 \\ \mathbb{C} & V_1 = V_2 \end{cases}$$

Corollary: $\mathrm{Hom}_G(V_1 \otimes \mathbb{C}^a, V_2 \otimes \mathbb{C}^b) = \mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^b) \otimes \mathrm{Hom}_G(V_1, V_2)$

Theorem: Let G be abelian, and (V, ρ) be irreducible representation. Then V is one-dimensional, i.e.

$$\rho \in \mathrm{Hom}(G, \mathbb{C}^\times)$$

Pf. We claim that for any $g_0 \in G$, $\rho(g_0) = \alpha(g_0)\, \mathrm{id}_V$.
Indeed, for any $g \in G$,

$$\rho(g_0)\rho(g) = \rho(g)\rho(g_0)$$

so $\rho(g_0) \in \mathrm{Hom}_G(V, V)$ so $\rho(g_0) = \alpha(g_0)$ by Schur's Lemma.
But then, ~~since~~ any subspace of V is invariant. V must be one-dimensional.

NB: • Working over $\mathbb{C}$ is important here. Example $G = SO(2) = U(1) = S^1$. has real representation

$$e^{i\theta} \longrightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2, \mathbb{R}) \subset GL(2, \mathbb{R})$$

this is irreducible.

• also note that this has two invariant endomorphisms $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

§2.2. Vollständige Zerlegbarkeit

<u>Def.</u> Eine Darstellung (V,ρ) von G heisst <u>unitär</u>, falls eine (positiv definite) <u>invariante</u> hermitesche Form existiert:

$$\langle \cdot,\cdot \rangle : V \times V \to \mathbb{C}$$

<u>Invarianz:</u> $\langle \rho(g)v, \rho(g)w \rangle = \langle v,w \rangle \quad \forall g,v,w$

<u>Beachte:</u> Falls solch ein inneres Produkt V mit V^* (dem Dualraum) identifiziert ($v \mapsto v^\# = \langle v, \cdot \rangle$; sesqui-linear!) so erhält man dadurch eine Identifikation

$$4(V^*,V) \underset{\langle\cdot,\cdot\rangle}{\cong} \mathrm{Hom}(V,V)$$

sesquilineare Abbildungen lineare Abbildungen

Falls jetzt V irreduzibel ist, so impliziert das Lemma von Schur, dass $\langle \cdot,\cdot \rangle \in \mathrm{Hom}_G(V \otimes V, \mathbb{C})$ bis auf einen skalaren Faktor eindeutig ist. Dies bedeutet noch <u>nicht</u> dass $V \cong V^*$ als Darstellung von G im Sinne der Eingangsbemerkung zu diesem Kapitel. ~~[illegible]~~

<u>Prop:</u> Sei jetzt G eine <u>endliche</u> Gruppe, und (V,ρ) eine Darstellung. Dann ist (V,ρ) unitär.

Corollary: Let V be a finite-dimensional representation of finite group G. Then there exist irreducible representations $V_1, \ldots V_k$ and multiplicities $a_1, \ldots a_k \in \mathbb{N}$ s.t.

$$V \cong \bigoplus_{i=1}^{k} V_i \otimes \mathbb{C}^{a_i} = \bigoplus_{i=1}^{k} V_i^{\oplus a_i}$$

(Complete reducibility)

This decomposition is unique (up to ordering and isomorphism).

Pf. For existence of decomposition, repeatedly apply Maschke's theorem. If

$$V = \bigoplus_{j=1}^{\ell} W_j \otimes \mathbb{C}^{b_j}$$

is another such decomposition, then by Schur's Lemma, since $id_V|_{V_i} : V_i \to V$ is a G-morphism the image must be irreducible representation isomorphic to $V_i \ldots$ $\blacksquare$

Note: • If G is not finite (and also not compact), complete reducibility is lost. E.g.

$$\mathbb{R} \ni a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in GL(2)$$

• ditto over fields other than $\mathbb{C}$.