Equivariant Localization

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1 Introduction

As a movivating example, we consider the integral

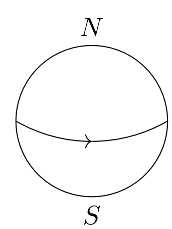
$$I(t) = \int_{S^2} \mathrm{dA}\, e^{itz}. \tag{1}$$

The stationary phase approximation has contribution from the north and south poles. Around the north pole,

$$\mathrm{dA} \sim dxdy, \quad z \sim 1 - \frac{1}{2}(x^2 + y^2), \tag{2}$$

and around the south pole,

$$\mathrm{dA} \sim -dxdy, \quad z \sim -1 + \frac{1}{2}(x^2 + y^2). \tag{3}$$



In the limit as $t \to \infty$, we get

$$I(t) \sim \frac{2\pi}{it} e^{it} + \frac{2\pi}{-it} e^{-it} = 4\pi \frac{\sin t}{t}. \tag{4}$$

This approximation is actually equal to the exact value of the integral,

$$I(t) = 2\pi \int_0^\pi d(\cos\phi) e^{it\cos\phi} = 2\pi \left(\frac{e^{it}}{it} - \frac{e^{-it}}{it} \right) = 4\pi \frac{\sin t}{t}. \tag{5}$$

The Duistermaat-Heckman formula (1982) states that certain kinds of integrals are exactly equal to their stationary phase approximation.

Berline and Vergne (1982) [4] and Atiyah and Bott (1984) [2] both showed that the DH formula is a consequence of a localization theorem for equivariant cohomology, which gives an exact result for integrals based on data around the fixed points of a symmetry.

Equivariant cohomology was invented by Borel (1959), and H. Cartan (1950). It is a cohomology theory that takes into account both a manifold M and the action of a group G.

2 Equivariant Differential

Consider the action of $G = S^1$ on an manifold M. The action is given by the flow of a vector field v. Define a differential

$$d_{S^1} = d - \iota_v. \tag{6}$$

This does not square to zero. However

$$d_{S^1}^2 = d^2 - d\iota_v - \iota_v d - \iota_v^2 = -L_v, \tag{7}$$

so if we restrict to S^1-invariant forms ($L_v \alpha = 0$), we get a chain complex and we can take its cohomology. We call a form α equivariantly-closed if $d_{S^1}\alpha = 0$ and equivariantly-exact if $\alpha = d_{S^1}\beta$ for some S^1 invariant form β. (Warning: this differential is not quite *the* differential for equivariant cohomology. We will see later that there is a factor of u missing, where u is the generator of the cohomology ring of BS^1)

Notice that equivariant forms are inhomogeneous

$$\alpha = \alpha_n + \alpha_{n-1} + \ldots + \alpha_0, \tag{8}$$

where α_i is a regular differential form of degree i. The condition $d_{S^1}\alpha = 0$ relates the pieces which are 2 degrees apart, i.e.

$$0 = d\alpha_n \tag{9}$$
$$0 = d\alpha_{n-1} \tag{10}$$
$$\iota_v \alpha_n = d\alpha_{n-2} \tag{11}$$
$$\iota_v \alpha_{n-1} = d\alpha_{n-3} \tag{12}$$
$$\vdots \tag{13}$$
$$\iota_v \alpha_2 = d\alpha_0 \tag{14}$$
$$\iota_v \alpha_1 = 0 \tag{15}$$
$$\iota_v \alpha_0 = 0 \tag{16}$$

The first and last conditions always hold by degree considerations.

We can define integration of inhomogeneous forms by integrating the piece of the correct degree:

$$\int_M \alpha := \int_M \alpha_n. \tag{17}$$

We immediately deduce an equivariant version of stokes theorem

$$\int_M d_{S^1}\alpha := \int_M d\alpha_{n-1} = \int_{\partial M} \alpha_{n-1} =: \int_{\partial M} \alpha, \tag{18}$$

so that the integral of an equivariant cohomology class is well defined on a manifold without boundary.

3 Equivariant Differential as a Supersymmetry

Locally, we use even and odd coordinates x_i and $\psi_i = dx_i$. Now an aribitrary form

$$\alpha = \alpha_n + \alpha_{n-1} + \ldots + \alpha_0 \tag{19}$$

can be written as a function of x_i and ψ_i

$$f(x,\psi) = f_n\psi_1 \cdots \psi_n + \ldots + f_0. \tag{20}$$

Integration of forms becomes superintegration, which is

$$\int_{\tilde{U}} \widetilde{\mathrm{dV}}\, f. \tag{21}$$

The super volume $\widetilde{\mathrm{dV}} = dx_1 \cdots dx_n d\psi_1 \cdots d\psi_n$ is well defined because of the Berezinian transformation rules. The equivariant differential d_{s^1} becomes an odd vector field Q given by

$$Q = \sum_i \left(\psi_i \frac{\partial}{\partial x_i} + v^i \frac{\partial}{\partial \psi_i} \right). \tag{22}$$

Supposing M has no boundary, Stokes' theorem $\int_M \; d_{S^1}\alpha = 0$ becomes

$$\int_{\tilde{M}} \widetilde{\mathrm{dV}}\, Q(f) = 0. \tag{23}$$

That is, the integral of a Q-exact function is zero. Note that in general (with $\partial M = 0$) we have

$$\int_{\tilde{M}} \widetilde{\mathrm{dV}}\, Q(f) = -\int_{\tilde{M}} \widetilde{\mathrm{dV}}\ \mathrm{Div}(Q) f. \tag{24}$$

One has to show that the Q-divergence is 0, or in other words, that the supervolume is Q-invariant. We could check this directly, or deduce it by relating to Stokes' theorem.

4 Localization Principle

The idea behind the example in the introduction is that the only contribution to the integral comes from the fixed points of the rotational symmetry on the sphere. Later, we will state prove the localization theorem, which gives an exact result for the integral. Before this, we will give some two simple arguments showing why there is no contribution to the integral the fixed points.

4.1 1st Localization Proof (Supersymmetric)

Away from the fixed locus, we can choose our coordinates so that $\frac{\partial}{\partial x_n} = v$. We then use a supersymmetric change of variables

$$x_i \mapsto x_i - \psi_i \psi_n. \tag{25}$$

This preserves the supervolume, and in these new coordinates,

$$Q = \psi_n \frac{\partial}{\partial x_n} - \frac{\partial}{\partial \psi_n} \tag{26}$$

$$Q^2 = -\frac{\partial}{\partial x_n}. \tag{27}$$

If f is Q-closed, then in particular $Q^2 f = 0$, which means that f doesn't depend on x_n, and futhermore, since $Q(f) = 0$, f doesn't depend on ψ_n either. It follows immediately that the integral

$$\int_{\tilde{M}} dx_1 \cdots d\psi_1 \cdots d\psi_n f = 0 \tag{28}$$

is 0, since the integrand has no ψ_n.

4.2 2nd Localization Proof (Poincare Lemma)

Let g be an invariant metric (i.e. $L_v g = 0$. We can always average over S^1 to get this). Let $\eta = g(v, -)$.

$$d_{S^1}\eta = -g(v, v) + d\eta. \tag{29}$$

$d_{S^1}^2 \eta = L_v \eta = 0$ by the invariance of g. Away from the fixed locus, $g(v, v) \neq 0$ so we can invert $d_{S^1}\eta$

$$\frac{1}{-g(v, v) + d\eta} = \frac{-1}{g(v, v)} \sum_k \left(\frac{d\eta}{g(v, v)} \right)^k. \tag{30}$$

The form

$$\Omega = \frac{\eta}{d_v \eta} \tag{31}$$

has the property that $d_v \Omega = 1$. Hence, away from the fixed locus, any equivariantly closed form α is equivariantly exact $\alpha = d_v(\Omega\alpha)$. It follows that its integral vanishes by the equivariant stokes theorem.

4.3 Localization Theorem

Either of the previous arguments show that for an equivariantly-closed form α,

$$\int_{M \setminus M^{S^1}} \alpha = 0 \tag{32}$$

(Actually, more properly we should say this is equal to an integral over ∂M^{S^1}). Now we would like to know what the intergral over the fixed locus M^{S^1} actually is. Let S^1 act on a manifold M of dimension $2n$ with isolated fixed points. With η as above, consider the closed equivariant form

$$\alpha e^{t(d_{S^1}\eta)}. \tag{33}$$

Its equviariant class is independent of t since

$$\frac{d}{dt}\alpha e^{t(d_{S^1}\eta)} = \alpha(d_{S^1}\eta)e^{t(d_{S^1}\eta)} = d_{S^1}\left(\alpha\eta e^{t(d_{S^1}\eta)}\right). \tag{34}$$

Around a fixed point we have

$$v = \sum_{i=1}^{n} \omega_i \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right), \tag{35}$$

$$g = \sum_{i=1}^{n} \left(dx_i^2 + dy_i^2 \right), \tag{36}$$

and

$$d_v\eta = -\sum_{i=1}^{n} \omega_i \left(x_i^2 + y_i^2 \right) + 2\sum_{i=1}^{n} \left(dx_i \wedge dy_i \right). \tag{37}$$

Now we calculate the integral as t goes to infinity.

$$\int_M \alpha e^{-t\omega_i\left(x_i^2+y_i^2\right)} e^{2t\sum_{i=1}^n (dx_i \wedge dy_i)} = \alpha_0 \prod_{i=1}^{n} \frac{2\pi}{\omega_i} + O(1/t). \tag{38}$$

We can do the same thing around every fixed point. Since the integral is independent of t, we compare with $t = 0$, and get the localization theorem

$$\int_M \alpha = (2\pi)^n \sum_{p \in M^{S^1}} \frac{\alpha_0(p)}{\prod_{i=1}^n \omega_{i,p}}. \tag{39}$$

After we define equivariant cohomology in the next section, we can state this result as

$$\int_M \alpha = \sum_p \frac{i_p^*\alpha}{e(N_p)} \tag{40}$$

where $e(N_p)$ is the equivariant euler class of the normal bundle of a fixed point.

Returning to introductory example: Let ω be the standard symplectic form on S^2. The height function z is actually the hamiltonian for the rotational vector field ∂_θ. If follows that

$$d_{S^1}(z + \omega) = dz - \omega(v, -) = 0. \tag{41}$$

The exponential

$$\frac{1}{it} e^{it(z+\omega)} \tag{42}$$

is also equivariantly closed. The top degree piece is dAe^{itz} and the zeroth degree piece is the stationary phase approximation. The generalization of this to any symplectic manifold and any hamiltonian f is the DH formula (for example, see [2]).

5 Equivariant Cohomology

The equivariant differential $d - \iota_v$ was a slight simplification which ignores some information about the S^1 action. We now give an overview of the full theory. Also, we might as well consider any Lie group G. There are basically 3 models of equivariant cohomology. A very thorough explanation can be found in [6].

The simplest idea for an equivariant cohomology would be the cohomology of the quotient M/G. It turns out, however, that this isn't quite the right thing consider. For example, in the case of the S^1 action, the quotient is a line, which is contractible, and the cohomology of a line is trivial.

Instead, we consider the space

$$M_G = M \times_G EG \tag{43}$$

where EG is the universal G bundle over the classifying space $BG = EG/G$, i.e. EG is a contractible space on which G acts freely. Then we define

$$H^*_G(M) = H^*(M_G). \tag{44}$$

This has the nice property that if G acts freely, then

$$H^*(M_G) = H^*(M/G \times EG) = H^*(M/G), \tag{45}$$

and if G acts trivially

$$H^*(M_G) = H^*(M \times EG/G) = H^*(M) \otimes H^*(BG). \tag{46}$$

In particular, the equivariant cohomology of a point $H^*_G(p)$ is $H^*(BG)$. Since every space maps to a point, the equivariant cohomology groups are all $H^*(BG)$ modules.

Example: Let $G = S^1$. The universal S^1 bundle is $EG = S^\infty$, and the classifying space is $BG = EG/S^1 = CP^\infty$ which can be seen by taking the limit $CP^n = S^{2n+1}/S^1$. Notice that S^∞ is contractible. The cohomology of the classifying space is in this case

$$H^*(BS^1) = H^*_{S^1}(p; \mathbb{C}) = H^*(CP^\infty; \mathbb{C}) = \mathbb{C}[u] \tag{47}$$

We would like some kind of deRham theory for equivariant cohomology. We should start with the space of forms

$$\Omega^*(M) \otimes \Omega^*(EG). \tag{48}$$

However, EG is probably infinite dimensional, so it's not clear at first what to use. To describe $\Omega^*(EG)$, Cartan uses the Weil aglebra

$$W = \wedge\mathfrak{g}^* \otimes S\mathfrak{g}^*. \tag{49}$$

We denote the generators of $\wedge\mathfrak{g}^*$ by θ^a (of degree 1), and the generators of $S\mathfrak{g}^*$ by u^a (of degree 2). G acts on $\mathfrak{g}^*$ via the coadjoint action, and hence it acts on W. The infinitesimal action is

$$\begin{aligned}
L_a\theta^b &= -C^b_{ac}\theta^c \\
L_a u^b &= -C^b_{ac}u^c \\
d\theta^i + \frac{1}{2}C^i_{jk}\theta^j\theta^k &= u^i \\
du^i &= C^i_{jk}u^j\theta^k \\
\iota_a\theta^b &= \delta^b_a \\
\iota_a u^b &= 0
\end{aligned} \tag{50}$$

This action is consistent with the relations between d, ι, L (these operators form a super-Lie-algebra, and all the actions above can be derived by starting only with a few, once again see [6], especially chapter 3). Notice that the θ^a act like connection forms, and u^a act like curvature forms.

Now that we can describe the forms on $M \times EG$, we need to describe the forms which descend to the quotient by S^1. These are given by

$$(\Omega^*(M) \otimes \Omega^*(EG))_{bas}. \tag{51}$$

where basic forms are those which are horizontal ($\iota_a\alpha = 0$) and invariant ($L_a\alpha = 0$). So we arrive at the Weil model of equivariant cohomology, that is the cohomology of

$$[(\Omega^*(M) \otimes W)_{bas}\, , d] \tag{52}$$

The Weil model is equivalent to the simpler Cartan Model

$$\left[(\Omega^*(M) \otimes S\mathfrak{g}^*)^G, d_C\right] \tag{53}$$

where

$$d_C = d - \sum u_a\iota_a. \tag{54}$$

In particular, for $G = S^1$, has dimension 1, so we get

$$d_C = d - u\,\iota_v \tag{55}$$

(where we write v for the vector field v on M instead of the label a of the element in $\mathfrak{g}^*$). Notice that since u_a has degree 2, this differential is actually degree 1 (unlike the simplified version we had earlier).

For M a point we have

$$H^*_{S^1}(p; \mathbb{C}) = \mathbb{C}[u] \tag{56}$$

Generalizing to any G, noticing that d_C acts trivially on $(S\mathfrak{g}^*)^G$, we get

$$H^*(BG) = H_G^*(pt) = (S\mathfrak{g}^*)^G. \tag{57}$$

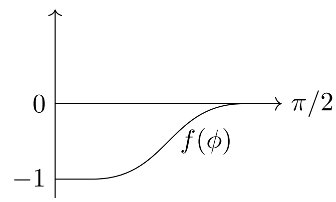
For $G = GL(n)$, the invariants $(S\mathfrak{g}^*)^G$ are the coefficients of t in

$$\det(t - A) \tag{58}$$

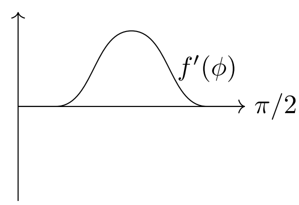
where $A \in \mathfrak{gl}(n)$.

6 Example: $H^*_{S^1}(S^2)$

Let's consider S^2 with the standard S^1 action. Let $f(\phi)$ be a smooth step function from -1 to 0 with support in $[0, \pi/2)$.



It's derivative is a bump function with integral 1.



The form

$$df d\theta - uf \tag{59}$$

is equivariantly closed. Similarly we can construct an equivariant closed form with support in the southern hemisphere. These generate the equivariant cohomology ring.

We can compare with the topological picture. Cover the sphere by two hemispheres. The intersection is S^1 and since the action here is free,

$$H^*_{S^1}(S^1) = H^*(S^1/S^1) = H^*(pt) = 1 \tag{60}$$

The northern and southern hemisphere each retract to a point which has equivariant cohomology $\mathbb{C}[u]$, so we have

$$H^*_{S^1}(S^2) = \mathbb{C}[x, y]/(xy) \tag{61}$$

Notice that if i is the inclusion of the north (or south) pole,

$$i^*(df d\theta - uf) = u. \tag{62}$$

7 Normal Bundle and Euler Class

If a bundle $P \to M$ comes with a G action that commutes with the projection, we get an equivariant bundle $P_G \to M_G$.

$$\begin{array}{c} P \times_G EG \\ \big\downarrow \\ M \times_G EG \end{array}$$

If $G = S^1$, the normal bundle N_p of an isolated fixed point p carries a natural S^1 action. Since a point has non-trivial equivariant cohomology, we can expect a non-trivial equivariant euler class $e_{S^1}(N_p) \subset BS^1$.

The normal bundle N_p is really just a vector space V, and the S^1 action decomposes

$$V = V_1 \oplus \cdots \oplus V_n \tag{63}$$

where V_i is 2-dimensional and S^1 acts with weight a_i. We can see that for any k,

$$\begin{array}{c} (V_1 \oplus \cdots \oplus V_n) \times_{S^1} S^{2k+1} \\ \big\downarrow \\ p \times_{S^1} S^{2k+1} \end{array} \tag{64}$$

is the bundle

$$\begin{array}{c} \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n) \\ \big\downarrow \\ CP^k \end{array}$$

which has euler class

$$(ua_1) \cdots (ua_n) = u^n (a_1 \cdots a_n) \in H^*(CP^k) = \mathbb{C}[u]/u^{k+1}. \tag{65}$$

Taking the k to infinity, we find the equivariant euler class of the normal bundle to a fixed point is

$$e_{S^1}(N_p) = u^n (a_1 \cdots a_n) \in H^*_{S^1}(p) = \mathbb{C}[u] \tag{66}$$

Rederivation of Localization Theorem: If $Q \subset M$ is a submanifold of codimension q, then there is a pushforward map

$$(i_Q)_* : H^*_{S^1}(Q) \to H^{*+q}_{S^1}(M) \tag{67}$$

the composition

$$(i_Q)^* \circ (i_Q)_* \alpha = \alpha \wedge e(N_Q) \tag{68}$$

If Q is the fixed locus of S^1, by localizating to $\mathbb{C}(u)$, we can invert $e(N_Q)$. We get an isomorphism

$$\frac{1}{e(N_Q)} (i_Q)^* : H^*_{S^1}(M) \to H^{*-q}_{S^1}(Q) \tag{69}$$

which is inverse to

$$(i_Q)_* : H^*_{S^1}(Q) \to H^{*+q}_{S^1}(M). \tag{70}$$

Using this isomorpism, the integral becomes

$$\int_M \alpha = \int_M (i_Q)_* \frac{(i_Q)^*\alpha}{e(N_Q)} = \int_Q \frac{(i_Q)^*\alpha}{e(N_Q)}, \tag{71}$$

which is just a sum for Q a collection of points.

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