Seminar on Supersymmetry in Geometry and Quantum Physics

Supersymmetric Lattice Models

Talk by Benjamin Freist

WiSe 2015/2016
Scribe: Sebastian Nill

1. Introduction/Motivation

Lattice models are at the basis of understanding transport in materials where still lots of things cannot be explained properly, e.g. high temperature superconductors, the fractional quantum Hall effect and more. Additional symmetry, e.g. supersymmetry, can simplify the computation in the lattice models.

Another important point is to understand supersymmetric field theories via their discretised form. The problem here is that the discretisation breaks the Lorentz invariance and thus it is hard to maintain the supersymmetry. A solution to this problem can be to start with a manifest supersymmetric lattice model and consider its continuum theory.

We are following chapter 1 to 4 of the PhD thesis *A supersymmetric model for lattice fermions* by Liza Huijse [**1**]. See also her slides of a talk given at KITP [**2**] and the paper *Lattice models with N=2 supersymmetry* by J. de Boer, P. Fendley, K. Schoutens [**3**].

2. The model

In this section we will construct a supersymmetric lattice model. We begin with some notation. By c_i we denote the operator that annihilates a fermion on

site i in the lattice. Its adjoint $c_i^\dagger$ creates a fermion on site i. They satisfy the anti-commutation relations

$$\{c_i^\dagger, c_j\} = \delta_{ij} \ , \quad \{c_i, c_j\} = 0 = \{c_i^\dagger, c_j^\dagger\} \ . \tag{2.1}$$

The last relation gives $(c_i^\dagger)^2 = 0$ and thus the Pauli exclusion principle. The operator $n_i = c_i^\dagger c_i$ detects a fermion on site i if present. With these definitions the Hamiltonian of the famous Hubbard model is given by

$$H = -t \sum_{i,<i,j>,\sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \ , \tag{2.2}$$

where $\sigma = \uparrow, \downarrow$ denotes the spin of the fermion. The notation $<i,j>$ in the first sum means that only neighboring sites j of the site i appear in the sum. Due to the first term in the Hamiltonian a fermion can hop to a neighboring site. Hence, this term is called kinetic. The second term describes a repulsive interaction of fermions of different spin on the same site. For low particle density and weak interactions $U << t$, the system is dominated by the kinetic term and behaves like a Fermi liquid. If the interactions are strong $U >> t$ and the lattice has on average one fermion per site, half-filled system, we get a Mott insulator. Apart from these special limits, it is difficult and in most cases not yet possible to solve the Hubbard model.

In order to get exact solutions of a lattice model, the approach we take is based on two simplifications. First, we will consider only spinless fermions. The second simplification is to work with a supersymmetric lattice model. In order to construct such a SUSY model, we call lattice configurations consisting of an odd or even number of fermions fermionic or bosonic, respectively. As the supercharge operator Q should induce a symmetry between fermionic and bosonic states, a natural choice for Q is

$$Q = \sum_i c_i^\dagger \ , \quad Q^\dagger = \sum_i c_i \ , \tag{2.3}$$

such that Q increases the fermion number by one and $Q^\dagger$ decreases it by one. The fermion number operator is given by

$$F = \sum_i n_i = \sum_i c_i^\dagger c_i \ , \qquad [F, Q] = Q \ , \quad [F, Q^\dagger] = -Q^\dagger \ . \tag{2.4}$$

where we have used the anti-commutation relations (2.1) in the last two equations. Using those relations again, we get the nilpotency of the supercharges

$$\{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\} \ , \quad H = \{Q^\dagger, Q\} = L \ , \tag{2.5}$$

where L is the total number of sites of our lattice. Unfortunenately, this approach yields a trivial Hamiltonian $H = L$ without any dynamics. For a more interesting model we impose a hard-core constraint on the fermions. This means we exclude states with neighboring fermions from the model. For that reason we introduce

$$P_{<i>} = \prod_{<i,j>} (1 - c_j^\dagger c_j) \ . \tag{2.6}$$

This self-adjoint operator $P_{<i>}$ projects onto states with unoccupied sites adjacent to site i. Note that $P_{<i>}$ commutes with c_j, if i and j are not adjacent, in particular if $i = j$. By substituting c_i with $c_i P_{<i>}$ and hence $c_i^\dagger$ with $c_i^\dagger P_{<i>}$ in the definition of the supercharges (2.3), we incorporate the hard-core constraint into the model.

$$Q = \sum_i c_i^\dagger P_{<i>} \ , \quad Q^\dagger = \sum_i c_i P_{<i>} \ . \tag{2.7}$$

Because of $[c_i, P_{<i>}] = 0$, the definition of the number operator F and its commutation relations with the supercharges (2.4) remain valid. Now the Hamiltonian reads

$$H = \{Q^\dagger, Q\} = \underbrace{\sum_{i,<i,j>} P_{<i>} c_i^\dagger c_j P_{<j>}}_{H_{\text{kin}}} + \underbrace{\sum_i P_{<i>}}_{H_{\text{pot}}} \quad , \tag{2.8}$$

which follows from a simple calculation using the anti-commutation relations (2.1). The first term H_{kin} is of kinetic nature enabling a fermion at site j to hop to the adjacent site i. This hopping is only allowed if neighboring sites of i are unoccupied before hopping and neighboring ones of j are so after hopping. The potential term H_{pot} describes the interaction of fermions.

It follows a short digression on the Witten index we will need in the next section. Since our model is supersymmetric, all energy states have non-negative energy.

$$\langle\psi|H|\psi\rangle = \| Q|\psi\rangle \|^2 + \| Q^\dagger|\psi\rangle \|^2 \geqslant 0 \tag{2.9}$$

As we have seen in the previous talks, our Hilbert space decomposes into irreducible representations of the supersymmetry algebra generated by Q und $Q^\dagger$. All zero energy states are singletts, because equation (2.9) yields $Q|\psi\rangle = 0$ and $Q^\dagger|\psi\rangle = 0$ for a zero energy state $|\psi\rangle$. The positive energy states transform as dubletts given by $\{|\psi\rangle\,, Q|\psi\rangle\}$ for a positive energy state $|\psi\rangle$ with $Q^\dagger|\psi\rangle = 0$. This special property of supersymmetric theories can be used to get control over the Witten index as follows. First, the Witten index W is defined as

$$W = \mathrm{Tr}[(-1)^F \mathrm{e}^{-\beta H}] = \sum_k (-1)^{f_k} \mathrm{e}^{-\beta E_k} \ , \tag{2.10}$$

where β is any non-negative real number (inverse temperature) and the sum is taken over an eigenbasis $\{|\psi_k\rangle\}_k$ with respect to the Hamiltonian H as well as the number operator F, simultaneously, with eigenvalues $H|\psi_k\rangle = E_k\,|\psi_k\rangle$ and $F|\psi_k\rangle = f_k\,|\psi_k\rangle$. This is possible since $[H, F] = 0$. It can also be achieved that $Q^\dagger|\psi_k\rangle = 0$ for all k. As seen above, any $|\psi_k\rangle$ with positive energy $E_k > 0$ pairs with $|\psi_m\rangle = Q|\psi_k\rangle$ having the same energy $E_m = E_k$ but the fermion number $f_m = f_k + 1$. Thus, only zero energy states contribute to the sum in the Witten index. Hence, W does not depend on β. Choosing $\beta = 0$ we get

$$W \ = \ \dim \left\langle \begin{array}{c} \text{bosonic zero} \\ \text{energy states} \end{array} \right\rangle \ - \ \dim \left\langle \begin{array}{c} \text{fermionic zero} \\ \text{energy states} \end{array} \right\rangle \ = \ \sum_k (-1)^{f_k} \ . \tag{2.11}$$

On the one hand, the right-hand side of this formula can be employed to easily compute the Witten index. We just need to know how many states there are of given fermion number. On the other hand, it provides us with a lower bound for the number of zero energy states, namely $|W|$. Consequently, if $W \neq 0$, the model has ground states at zero energy.

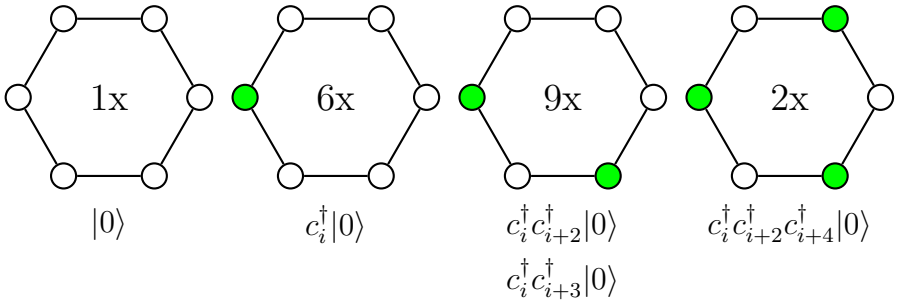
Now, let us consider an 1-dim lattice with L sites and periodic boundary conditions, so a closed chain of length L. We label the sites by $i = 1, \ldots, L$. Indices beyond this range should be thought of modulo L. If we set $P_i = 1 - n_i$, the projection operator simplifies to $P_{<i>} = P_{i+1}P_{i-1}$. Now the Hamiltonian becomes

$$H = \underbrace{\sum_i P_{i+2}(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i)P_{i-1}}_{\substack{= H_{\text{kin}} \\ \text{nearest nb hopping term}}} + L \underbrace{- 2F}_{\substack{\text{chemical} \\ \text{potential}}} + \underbrace{\sum_i n_{i-1}n_{i+1}}_{\substack{\text{next-nearest} \\ \text{nb repulsion}}} \tag{2.12}$$

Here, we can see that the potential part of the Hamiltonian consists of a constant L, a chemical potential $\mu = 2$ and a next-nearest neighbor repulsion term of strength $V_2 = 1$. Due to the hard-core constraint, the nearest neighbor repulsion strength is $V_1 = \infty$. With the notation of the Hubbard model in mind, the hopping term contributes with strength $t = 1$ to the dynamics.

3. The 6-site chain

In this section, we set $L = 6$ and solve our lattice model explicitly for the 6-site chain. In this case, the combinatorics to get all particle configurations are still accessible. Keeping the Pauli exclusion principle and the hard-core constraint in mind, we get the following allowed configurations clustered by fermion number.



For fermion number $f = 2$ there are six states of the form $c_i^\dagger c_{i+2}^\dagger|0\rangle$ for $i = 1, \ldots, 6$, and three states of the form $c_i^\dagger c_{i+3}^\dagger|0\rangle$ for $i = 1, 2, 3$. The Hilbert space has got $1+6+9+2 = 18$ dimensions. At this stage, all the information we need to compute the Witten index with formula (2.11) is available.

$$W = 1 - 6 + 9 - 2 = 2 \tag{3.1}$$

Thus, we can already say that our model has at least $|W| = 2$ zero energy states. Let us find eigenstates of the Hamiltonian (2.12). Because of $c_i |0\rangle = 0$, the empty lattice state $|0\rangle$ is an eigenstate with energy $E = L = 6$. If one acts with the Hamiltonian on the two 3-fermion states $|135\rangle$ and $|246\rangle$, one sees that the hopping term does not contribute due to the hard-core constraint and both states are already eigenstates with energy $E = 6 - 6 + 3 = 3$. We make the following educated guess for the eigenstates with one fermion

$$|a_n\rangle = |1\rangle + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}n} |2\rangle + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}2n} |3\rangle + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}3n} |4\rangle + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}4n} |5\rangle + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}5n} |6\rangle \ , \tag{3.2}$$

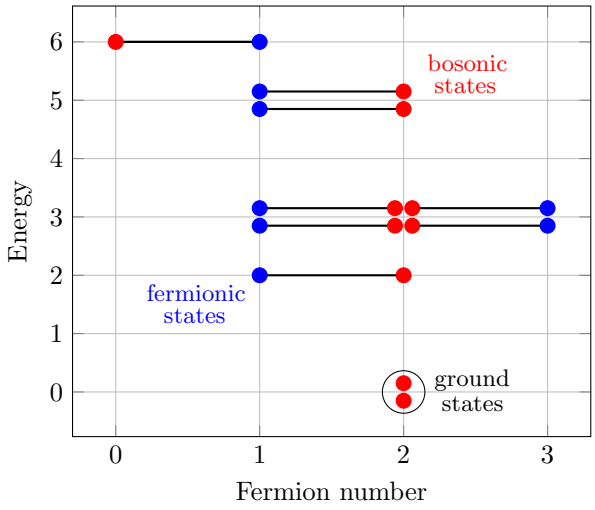
for $n = 0, \ldots, 5$. One easily convinces oneself that the following holds

$$H |a_n\rangle = \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{3}n} + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}n} + 6 - 2\right) |a_n\rangle = \left(4 + \cos\frac{\pi n}{3}\right) |a_n\rangle \ . \tag{3.3}$$

Therefore, the model admits one 1-fermion eigenstate with energy $E = 2$, two with $E = 3$, two with $E = 5$, and finally one with $E = 6$. Furthermore, it holds

$$Q |0\rangle = |a_0\rangle \quad \text{thus} \quad Q |a_0\rangle = 0 \ , \qquad \text{and} \qquad Q^\dagger |a_{n\neq 0}\rangle = 0 \ . \tag{3.4}$$

Thus, $|0\rangle$ and $|a_0\rangle$ are superpartners. Since the remaining five 1-fermion states $|a_{n\neq 0}\rangle$ have positive energy, they must find five superpartners among the 2-fermion states, namely $Q |a_{n\neq 0}\rangle$. Something similar is true for the two 3-fermion eigenstates also exhibiting positive energy. They have to pair up with the two 2-fermion states $Q^\dagger |135\rangle$ and $Q^\dagger |246\rangle$. Hence, we are left with $9 - 5 - 2 = 2$ states with fermion number $f = 2$ missing superpartners because all states with other fermion number have already formed superpairs. Consequently, these leftover 2-fermion states possess zero energy and our model owns two zero energy ground states in agreement with the Witten index $W = 2$ calculated in (3.1). The following diagram summarizes our results and shows in which way the eigenstates pairing up under supersymmetry.



4. More general setups

Let us come back to the general closed L-site chain and determine its Witten index, since the index is still accessible, but gives us in numerous cases a sufficient criterion for the existence of zero energy ground states. For the computation we use a modified version of the partition function.

$$Z(z) = \mathrm{Tr}\left(z^F \mathrm{e}^{-\beta H}\right) = \sum_k z^{f_k} \mathrm{e}^{-\beta E_k} \ , \tag{4.1}$$

where as above the sum is taken over an eigenbasis $\{|\psi_k\rangle\}_k$ with respect to the Hamiltonian H and the number operator F with eigenvalues E_k and f_k, respectively. Consider the limit $\beta \to 0$ to get rid of the energy dependence.

$$Z(z) = \mathrm{Tr}\left(z^F\right) = \sum_k z^{f_k} \tag{4.2}$$

Two special choices of z yield $Z(1) =$ dimension of Hilbert space and $Z(-1) = W$ Witten index. Dropping the condition that $\{|\psi_k\rangle\}_k$ should form an eigenbasis for H, we can demand that every $|\psi_k\rangle$ describes a pure particle configuration. Hence, it makes sense to denote by n_{ki} the eigenvalue of n_i to the eigenstate $|\psi_k\rangle$.

$$n_{ki} = \begin{cases} 1 & \text{if site } i \text{ is occupied in state } |\psi_k\rangle \\ 0 & \text{if site } i \text{ is unoccupied in state } |\psi_k\rangle \end{cases} \ , \qquad f_k = \sum_i n_{ki} \tag{4.3}$$

With this notation we rewrite

$$Z(z) = \sum_k \prod_i z^{n_{ki}} = \mathrm{Tr}\, \mathcal{T}^L \ , \tag{4.4}$$

where the transfer matrix $\mathcal{T}$ is defined as

$$\mathcal{T}_{rs} = (1 - \delta_{2,r+s})\, z^r \ , \quad r,s = 0,1 \quad \text{or equivalently} \quad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ z & 0 \end{pmatrix} \ . \tag{4.5}$$

If we think of the basis vectors $(0,1)$ and $(1,0)$ to encode the information that a site is occupied and unoccupied, respectively, the transfer matrix $\mathcal{T}$ describes the state the next site is in. If the site is unoccupied, we act with $\mathcal{T}$ on $(1,0)$ and get $(1,z)$. This means the state for the next site is a combination of unoccupied and occupied. The z tracks the fermions. In this manner, the exponent of z gives the number of fermions. If the site is occupied, acting with $\mathcal{T}$ on $(0,1)$ gives $(1,0)$. This means the next site has to be unoccupied as demanded by the hard-core constraint. The eigenvalues of the transfer matrix $\mathcal{T}$ are given by $\frac{1}{2}\left(1 \pm \sqrt{1+4z}\right)$. Therefore,

$$Z(z) = \mathrm{Tr}\, \mathcal{T}^L = \left(\frac{1+\sqrt{1+4z}}{2}\right)^L + \left(\frac{1-\sqrt{1+4z}}{2}\right)^L \ . \tag{4.6}$$

With $z = -1$ we get the Witten index

$$W = \left(\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\right)^L + \left(\frac{1}{2} - \frac{\mathrm{i}}{2}\sqrt{3}\right)^L = \mathrm{e}^{\mathrm{i}\frac{\pi L}{3}} + \mathrm{e}^{-\mathrm{i}\frac{\pi L}{3}} = 2\cos\left(\frac{\pi L}{3}\right) \tag{4.7}$$

$$= \begin{cases} 2 & \text{if} \quad L \equiv 0 \mod 6 \\ 1 & \text{if} \quad L \equiv 1 \mod 6 \quad \text{or} \quad L \equiv 5 \mod 6 \\ -1 & \text{if} \quad L \equiv 2 \mod 6 \quad \text{or} \quad L \equiv 4 \mod 6 \\ -2 & \text{if} \quad L \equiv 3 \mod 6 \end{cases}$$

In all cases $|W| \geqslant 1$ holds. Thus, the closed L-site chain model has always at least one zero energy ground state.

For the open L-site chain, similar considerations yield

$$Z(z) = \sum_{r,s=0}^{1} \left(\mathcal{T}^{L-1}\right)_{rs} c_s \ , \tag{4.8}$$

where $c = (1, z)$. For $z = -1$ this gives the Witten index

$$W = -\frac{2}{\sqrt{3}} \sin\left(\frac{\pi(L-1)}{3}\right) \tag{4.9}$$

$$= \begin{cases} 1 & \text{if} \quad L \equiv 0 \mod 6 \quad \text{or} \quad L \equiv 5 \mod 6 \\ 0 & \text{if} \quad L \equiv 1 \mod 6 \quad \text{or} \quad L \equiv 4 \mod 6 \\ -1 & \text{if} \quad L \equiv 2 \mod 6 \quad \text{or} \quad L \equiv 3 \mod 6 \end{cases}$$

In the cases $L \equiv 1 \mod 6$ and $L \equiv 4 \mod 6$, we cannot say from the computation of the Witten index, whether there are any zero energy ground states.

When considering lattices of dimension $D > 1$, it is much harder to obtain analytic expressions for the Witten index. But at least for 2-dimensional lattices some generic features are known. The most important one is the exponential growth of the Witten index with the size of the lattice leading to a non-vanishing entropy at zero temperature $T = 0$. On a square lattice, this exponential growth can be seen as the number of zero energy ground states is connected to the number of tilings of the lattice.

5. Link to topology

The hard-core fermion model defined in section 2 lives on a lattice, but obviously it can easily be generalized to an arbitrary graph by making the same definitions. In order to determine the zero energy states, we need the notion of an independence complex Σ. Every element σ in Σ is a subset of the vertex set S of the underlying graph with the condition that any two vertices in σ are not adjacent. Due to the hard-core constraint, every σ describes a fermion configuration on the graph and we have actually a one-to-one correspondence between Σ and the set of

all fermion configurations. Let C_n be the subspace of the Hilbert space spanned by all subgraphs $\sigma \in \Sigma$ with exactly n vertices. As seen in section 2, the supercharge Q maps from C_n to C_{n+1} and is nilpotent, i.e. $Q^2 = 0$. Therefore, we can compute the cohomology H^*_Q of the cochain complex (C_*) with coboundary map Q. In a similar way, the supercharge $Q^\dagger$ leads to a chain complex, namely (C_*) with boundary map $Q^\dagger : C_n \to C_{n-1}$, and one can define its homology.

$$C_0 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_1 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_2 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_3 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} \cdots$$

The zero energy ground states are related to the cohomology of Q or equivalently the homology of $Q^\dagger$.

$$\begin{array}{ccccc} \left\langle \begin{array}{c} \text{zero energy states} \\ \text{with particle number } n \end{array} \right\rangle & \cong & H^n_Q & = & \dfrac{\ker Q : C_n \longrightarrow C_{n+1}}{\operatorname{im} Q : C_{n-1} \longrightarrow C_n} \\ |\psi\rangle & \longmapsto & & & [\ |\psi\rangle\ ] \end{array} \tag{5.1}$$

Thus, the Witten index is the Euler characteristic χ associated to the cohomology.

$$W \;=\; \chi \;=\; \sum_n (-1)^n \dim H^n_Q \tag{5.2}$$

For the actual computation of the cohomology we use the tic-tac-toe lemma in the following way. First, we partition the vertex set S in a smart way - what smart means we will see later - into two subsets S_1 and S_2, i.e.

$$S = S_1 \cup S_2 \qquad \text{with} \qquad S_1 \cap S_2 = \varnothing \ . \tag{5.3}$$

Accordingly, we can split the supercharge Q into Q_1 and Q_2, such that

$$Q = Q_1 + Q_2 \qquad \text{where} \qquad Q_m = \sum_{i \in S_m} c_i^\dagger P_{<i>} \ , \quad m = 1, 2 \ . \tag{5.4}$$

By $K_{p,q}$ we denote the subspace of C_{p+q} spanned by all particle configurations $\sigma \in \Sigma$, such that exactly p fermions of σ lie in S_1 and q in S_2.

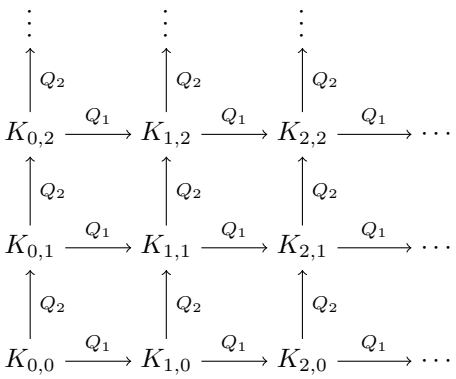
$$\begin{aligned} K_{p,q} &= \langle\ \sigma \in \Sigma \ \mid\ \#(\sigma \cap S_1) = p\ ,\ \#(\sigma \cap S_2) = q\ \rangle \ \subseteq\ C_{p+q}\ , \\ C_n &= \bigoplus_{p+q=n} K_{p,q} \quad , \quad Q_1 : K_{p,q} \to K_{p+1,q} \quad , \quad Q_2 : K_{p,q} \to K_{p,q+1}\ . \end{aligned} \tag{5.5}$$

Because of the anti-commutation relations (2.1) of the c_i, we get $\{Q_1, Q_2\} = 0$. Hence, $(K_{*,*})$ with the two differentials Q_1 and Q_2 forms a double complex. For fixed column p, we can compute the cohomology of the column complex $(K_{p,*})$ with differential Q_2 and obtain $H^q_{Q_2}(K_{p,*})$. For fixed row q, the horizontal map Q_1 induces differentials for the complex $(H^q_{Q_2}(K_{*,*}))$. Its cohomology is given by

$$H^{p,q}_{12} := H^p_{Q_1}(H^q_{Q_2}(K_{*,*}))\ . \tag{5.6}$$

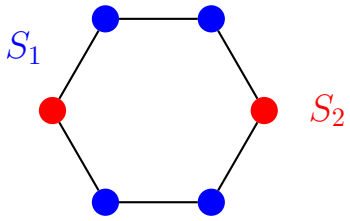
Now, the *tic-tac-toe lemma* states that if there exists a row q, such that $H^{*,*}_{12}$ is non-vanishing only in this row q, i.e. $H^{p,r}_{12} = 0$ for all p and $r \neq q$, then

$$H^{p+q}_Q \cong H^{p,q}_{12} = H^p_{Q_1}(H^q_{Q_2})\ . \tag{5.7}$$

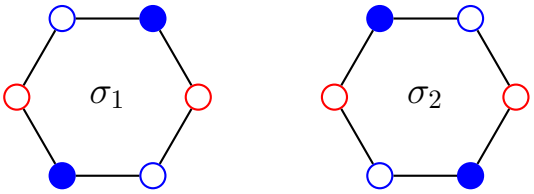


The partition (S_1, S_2) of S should be chosen in such a smart way that this sufficient condition for the tic-tac-toe lemma is fulfilled.

As an example let us again consider the closed 6-site chain from section 3. For the vertex set $S = 1, \ldots, 6$ we choose the partition $S_1 = 2, 3, 5, 6$ and $S_2 = 1, 4$.



We have to determine H_{Q_2}. Assume at first a particle configuration on S_1 such that both sites adjacent to one of the sites in S_2 are empty. Let us suppose this is satisfied for site $1 \in S_2$. If site 1 is not occupied, Q_2 can create a particle there. Otherwise, the configuration is in the image of Q_2. Thus, we obtain $H_{Q_2} = 0$ in those cases. The analogue is true for site $4 \in S_2$. Hence, only two configurations σ_1 and σ_2 remain



Due to the hard-core constraint, both lie in the kernel of Q_2. As they do not contain any particles in S_2, they cannot be in the image of Q_2. Therefore, both represent independent classes in H_{Q_2}. In summary we get

$$H_{Q_2}^{2,0} = \langle \sigma_1, \sigma_2 \rangle \qquad \text{and} \qquad H_{Q_2}^{p,q} = 0 \quad \text{for all} \quad (p,q) \neq (2,0) \; . \tag{5.8}$$

Consequently, the assumption in the tic-tac-toe lemma is satisfied and the lemma yields

$$H_Q^2 \cong H_{Q_1}^2(H_{Q_2}^{*,0}) = \langle \sigma_1, \sigma_2 \rangle \ , \tag{5.9}$$
$$H_Q^p \cong H_{Q_1}^p(H_{Q_2}^{*,0}) = 0 \qquad \text{for} \qquad p \neq 2 \ .$$

We see that there are two zero energy ground states with two fermions each, in agreement with the results obtained in section 3.

In order to *prove the tic-tac-toe lemma* we will construct maps

$$\Phi : H_{12}^{p,q} \longrightarrow H_Q^{p+q} \qquad \text{and} \qquad \Psi : H_Q^{p+q} \longrightarrow H_{12}^{p,q} \tag{5.10}$$

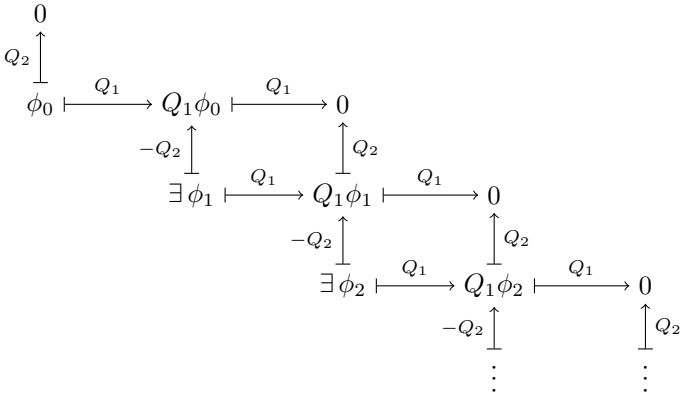
being mutually inverse. We begin with Φ. Pick a double class $[[\phi_0]] \in H_{12}^{p,q} = H_{Q_1}^p(H_{Q_2}^{*,q})$ represented by an element $\phi_0 \in K_{p,q}$. We have $[\phi_0] \in H_{Q_2}^{p,q}$ and $Q_1[\phi_0] = 0$. Hence,

$$Q_2\phi_0 = 0 \qquad \text{and} \qquad Q_1\phi_0 = -Q_2\phi_1 \quad \text{for some} \quad \phi_1 \in K_{p+1,q-1} \ . \tag{5.11}$$

Now, applying Q_2 to $Q_1\phi_1$ yields

$$\begin{aligned}
& Q_2Q_1\phi_1 = -Q_1Q_2\phi_1 = Q_1^2\phi_0 = 0 \\
\Rightarrow \quad & [Q_1\phi_1] \in H_{Q_2}^{p+2,q-1} = 0 \quad \text{by assumption} \\
\Rightarrow \quad & Q_1\phi_1 = -Q_2\phi_2 \qquad \text{for some} \quad \phi_1 \in K_{p+2,q-2} \\
\Rightarrow \quad & Q_2Q_1\phi_2 = -Q_1Q_2\phi_2 = Q_1^2\phi_1 = 0 \\
\Rightarrow \quad & [Q_1\phi_2] \in H_{Q_2}^{p+3,q-2} = 0 \quad \text{by assumption} \\
\Rightarrow \quad & Q_1\phi_2 = -Q_2\phi_3 \qquad \text{for some} \quad \phi_2 \in K_{p+3,q-3} \\
\Rightarrow \quad & \dots
\end{aligned} \tag{5.12}$$

Inductively, we get a series $\{\phi_k\}_{k=0,\dots,q}$ with $\phi_k \in K_{p+k,q-k}$ and $Q_1\phi_k = -Q_2\phi_{k+1}$.



We define $\phi := \sum_{k=0}^{q} \phi_k \in C_{p+q}$. Using an index shift and $Q_1\phi_k + Q_2\phi_{k+1} = 0$, we obtain

$$Q\phi = Q_1\phi + Q_2\phi = \sum_{k=0}^{q} (Q_1\phi_k + Q_2\phi_{k+1}) + Q_2\phi_0 = 0 \ . \tag{5.13}$$

Thus, ϕ defines a class in H_Q^{p+q} and we set $\Phi([\phi_0]) = [\phi]$.

In order to construct Ψ, let $[\phi]$ be a class in H_Q^{p+q} represented by $\phi \in C_{p+q}$. Because of $C_{p+q} = \bigoplus_{k=-p}^{q} K_{p+k,q-k}$, we can decompose ϕ in the following way

$$\phi = \sum_{k=-p}^{q} \phi_k \qquad \text{where} \quad \phi_k \in K_{p+k,q-k} \ . \tag{5.14}$$

As ϕ represents a class in H_Q^{p+q}, an index shift provides us with

$$0 = Q\phi = Q_1\phi + Q_2\phi = \sum_{k=-p-1}^{q} \underbrace{(Q_1\phi_k + Q_2\phi_{k+1})}_{\in K_{p+k+1,q-k}}$$

$$\Rightarrow \quad Q_1\phi_k = -Q_2\phi_{k+1} \qquad \text{for all} \quad k \tag{5.15}$$

Now, the idea is to modify ϕ within its class in H_Q^{p+q} such that $Q_2\phi_0 = 0$ and ϕ_0 describes a class in $H_{12}^{p,q}$. We start with ϕ_{-p}. Relation (5.15) yields for $k = -p-1$

$$\begin{aligned}
& Q_2\phi_{-p} = 0 \\
\Rightarrow \quad & [\phi_{-p}] \in H_{Q_2}^{0,p+q} = 0 \quad \text{by assumption} \\
\Rightarrow \quad & \phi_{-p} = -Q_2\psi_{-p} \qquad \text{for some} \quad \psi_{-p} \in K_{0,p+q-1} \\
\Rightarrow \quad & Q_2\tilde{\phi}_{-p+1} = Q_2\phi_{-p+1} - Q_1Q_2\psi_{-p} = Q_2\phi_{-p+1} + Q_1\phi_{-p} = 0 \quad \text{by} \quad (5.15) \\
& \qquad\qquad \text{where} \quad \tilde{\phi}_{-p+1} := \phi_{-p+1} + Q_1\psi_{-p} \in K_{1,p+q-1} \\
\Rightarrow \quad & [\tilde{\phi}_{-p+1}] \in H_{Q_2}^{1,p+q-1} = 0 \quad \text{by assumption} \\
\Rightarrow \quad & \tilde{\phi}_{-p+1} = -Q_2\psi_{-p+1} \qquad \text{for some} \quad \psi_{-p+1} \in K_{1,p+q-2} \\
\Rightarrow \quad & Q_2\tilde{\phi}_{-p+2} = Q_2\phi_{-p+2} - Q_1Q_2\psi_{-p+1} = Q_2\phi_{-p+2} + Q_1\phi_{-p+1} = 0 \quad \text{by} \quad (5.15) \\
& \qquad\qquad \text{where} \quad \tilde{\phi}_{-p+2} := \phi_{-p+2} + Q_1\psi_{-p+1} \in K_{2,p+q-2} \\
\Rightarrow \quad & \ldots
\end{aligned}$$

After p of these steps, we obtain $\tilde{\phi}_0 = \phi_0 + Q_1\psi_{-1} \in K_{p,q}$ and a series $\{\psi_k\}_{k=-p,\ldots,-1}$ with $\psi_k \in K_{p+k,q-k-1}$ such that $-\phi_{k+1} = Q_1\psi_k + Q_2\psi_{k+1}$. From this follows

$$Q\left(\sum_{k=-p}^{-1} \psi_k\right) = \underbrace{\sum_{k=-p-1}^{-2} (Q_1\psi_k + Q_2\psi_{k+1})}_{= -\sum_{k=-p}^{-1} \phi_k} + \underbrace{Q_1\psi_{-1}}_{\tilde{\phi}_0 - \phi_0} = \tilde{\phi} - \phi \tag{5.16}$$

where $\tilde{\phi} := \tilde{\phi}_0 + \sum_{k=0}^{q} \phi_k$ is the modified version of ϕ. Indeed, equation (5.16) tells us that $[\tilde{\phi}] = [\phi]$ as classes in H_Q^{p+q}. Moreover, we have

$$Q_2\tilde{\phi}_0 = 0 \qquad \text{and} \qquad Q_1\tilde{\phi}_0 = Q_1\phi_0 + Q_1^2\psi_{-1} = -Q_2\phi_1 \quad \text{by} \quad (5.15), \tag{5.17}$$

so $\tilde{\phi}_0$ represents a class in H_Q^{p+q}. Finally, we set $\Psi([\phi]) = [\tilde{\phi}_0]$. Obviously, Φ and Ψ are inverse to each other. It remains to check that Φ and Ψ are independent of the choices we made for representatives and preimages.

□

References

1. L. Huijse. *A supersymmetric model for lattice fermions*. http://www.lizahuijse.net/sites/default/files/Huijse_PhD_Thesis.pdf. PhD thesis, 2010.
2. L. Huijse. *A supersymmetric model for lattice fermions*. http://online.kitp.ucsb.edu/online/adscmt11/huijse/pdf/Huijse_AdScmt_KITP.pdf. Slides of a talk given at the conference "Holographic Duality and Condensed Matter Physics" at KITP, 2011.
3. J. de Boer, P. Fendley, K. Schoutens. *Lattice models with N=2 supersymmetry*. [hep-th/0210161] Phys.Rev.Lett. 90 (2003) 120402.