Sigma Models and Landau-Ginzburg Models in Supersymmetric Quantum Mechanics

Seminar on Supersymmetry in Geometry and Quantum Mechanics

Michael Bleher

11. November 2015

1. Revision

Recall that in general a Quantum Field Theory may be specified by the following data

- A **worldvolume** manifold $\mathcal{T}$
- A collection of **fields** $\Phi : \mathcal{T} \to M$ taking values in some target space M
- An **action** functional S from the space of field configurations to the field of real numbers $S : \{\Phi\} \to \mathbb{R}$
- A **quantisation procedure**, eg. canonical quantisation or path-integral quantisation and the fixing of amibiguities

In previous talks we considered QFTs with $\dim \mathcal{T} = 0$ and $\dim \mathcal{T} = 1$ and a supersymmetric collection of fields $\{\phi, \psi, \overline{\psi}\}$. In particular $\phi : \mathcal{T} \to \mathbb{R}$ is bosonic, whereas $\psi, \overline{\psi} : \mathcal{T} \to \mathbb{C}$ are fermionic (anticommuting) fields. As we have seen the case of a one-dimensional worldvolume is equivalent to supersymmetric quantum mechanics (SQM), where we identify the parametrisation of the worldline $\mathcal{T}$ with time t.

The main goal of this talk is to introduce one-dimensional supersymmetric sigma models along these lines, by essentially mapping the worldline $\mathcal{T}$ into a generic Riemannian manifold (M, g). To specify some main properties of such theories we will repeatedly make use of the following result that we encountered in the previous talk.

Theorem 1. *Suppose we are given a quantum theory with the following structure:*

- *a graded Hilbert space $\mathcal{H} = \bigoplus_p \mathcal{H}^p$, where the grading is with respect to the Fermion number operator F*

- *an action of the super-Poincaré algebra*[1]

$$\{Q,\overline{Q}\} = 2H \qquad [H,Q] = 0 = [H,\overline{Q}] \qquad \{Q,Q\} = 0 \tag{1}$$

  on this Hilbert space.

- *a mass gap in the spectrum of the Hamiltonian H, i.e. eigenvalues* $0 = E_0 < E_1 \leq \cdots$

Then the supersymmetric ground states are given by the Q-cohomology, i.e.

$$\mathcal{H}_0 = \bigoplus H^p(Q) = \bigoplus \frac{\ker\left(Q: \quad \mathcal{H}^p \ \to \mathcal{H}^{p+1}\right)}{\mathrm{im}\ \left(Q: \mathcal{H}^{p-1} \to \ \mathcal{H}^p\ \right)} \quad . \tag{2}$$

Further we can identify the Witten Index with the Euler-Characteristic of this cohomology

$$\mathrm{Tr}(-1)^F = \sum (-1)^p \dim H^p(Q)\ . \tag{3}$$

In section 2 we will apply this reasoning to Sigma Models in SQM and in doing so reveal a strong link to the geometry of its target space. We then go on to discuss Landau-Ginzburg Models in section 3, by considering additional structure on the target manifold.

2. Sigma Models

A sigma model is a QFT with a (bosonic) scalar field ϕ that maps to a smooth Riemannian target manifold (M, g) of dimension d. Put differently the bosonic field defines an embedding of the worldvolume $\mathcal{T}$ into the target $\phi(\mathcal{T}) \subset M$. The action of a sigma model is given by the volume of the embedding, namely the integral of the pullback metric, and a potential $V(\phi)$

$$S = \int_{\mathcal{T}} (\phi^* g - V(\phi)) = \int_{\mathcal{T}} \mathrm{d}t\ \left(\frac{1}{2} g_{AB}(\phi)\dot{\phi}^A\dot{\phi}^B - V(\phi)\right)\ . \tag{4}$$

Here – and later – we sloppily write $V(\phi)$ on the left hand side, where we really mean the top-dimensional form $V(\phi)\,\mathrm{d}t$ on $\mathcal{T}$ with values in M.

Eg. in the case of relativistic quantum mechanics the target is spacetime $\mathbb{R}^{1,3}$ and the embedding is the worldline of the particle $x^\mu(t)$ parametrised by proper time $t \in \mathcal{T} \simeq \mathbb{R}$. In analogy to this we will from now on refer to the bosonic maps as $x^i(t) \equiv x^i \circ \phi(t)$, where $\{x^i\}$ are local coordinates on the manifold M.

[1] We only consider 1-dim worldvolumes, which only have one Poincaré generator H corresponding to translations in the parameter $t \to t + c$.

2.1. Supersymmetric Action

In a supersymmetric sigma model we need to include fermionic maps $\psi(t), \overline{\psi}(t)$ in such a way that the bosonic part of the action is as above and the entire action is invariant under supersymmetry transformations of the fields. The most transparent way to do this is in terms of superspace and superfields, which makes supersymmetry manifest.

Thus let $\mathcal{T}^{1|2}$ be a supermanifold with 1 bosonic and 2 fermionic dimensions, (locally) given by the coordinates $(t, \Theta, \overline{\Theta})$. The automorphism group of $\mathcal{T}^{1|2}$ is generated by the super-Poincaré algebra (1), which also generates the supersymmetry transformations of the fields – as we have seen in a previous talk. Due to this fact supersymmetry transformations of $\{x, \psi, \overline{\psi}, F\}$ can be described by translations in superspace via their effect on the superfield[2]

$$\Phi(t, \Theta, \overline{\Theta}) = x(t) + \overline{\Theta}\psi(t) + \Theta\overline{\psi}(t) + \overline{\Theta}\Theta F(t) \ .$$

Thus it is easy to see that any coordinate independent superfield expression will be manifestly supersymmetric. In particular any polynomial of these superfields integrated over superspace is such an expression. Furthermore one can define SUSY-covariant derivatives D, $\overline{D}$, such that we can include derivatives of the superfields into these polynomials.

Finally one can see that the bosonic action (4) from above is exactly contained in the obvious and manifestly supersymmetric generalisation

$$S = \int_{\mathcal{T}^{1|2}} \Theta \left[\Phi^* g - V(\Phi)\right] = \int_{\mathcal{T}^{1|2}} \mathrm{d}t \mathrm{d}^2\Theta \left[\frac{1}{2} g_{ab}(\Phi)\, \overline{D}\Phi^a\, D\Phi^b - V(\Phi)\right] \ . \tag{5}$$

After a lengthy but straightforward calculation – carried out in detail in Appendix A – one finds

$$\begin{aligned} S = \int \mathrm{d}t \Bigg[& \frac{1}{2}\, g_{ab}(x)\, \partial_t x^a\, \partial_t x^b + \frac{i}{2}\, g_{ab}(x) \left(\overline{\psi}^a\, \nabla_t \psi^b - \nabla_t \overline{\psi}^a\, \psi^b\right) - \frac{1}{2} R_{abcd}\, \overline{\psi}^a \psi^b\, \overline{\psi}^c \psi^d \\ & - \frac{1}{2}\, g^{ab}\, \partial_a h(x)\, \partial_b h(x) - \nabla_a \partial_b h(x)\, \overline{\psi}^a \psi^b \Bigg] \ , \end{aligned} \tag{6}$$

where $\nabla_t \psi^i = \partial_t \psi^i + \Gamma^i_{jk}\, \partial_t x^j\, \psi^k$ is the pullback of the Levi-Civita connection ∇_a on (M, g), and the potential $h(x) = V(\Phi)|_{\Theta=\overline{\Theta}=0}$ is some polynomial in x. Note that we integrated out the auxiliary field F, i.e. replaced it by its equation of motion.

By construction this action is super-Poincaré invariant, which in particular includes the following SUSY transformations.

$$\begin{aligned} \delta x^i &= \epsilon \overline{\psi}^i - \bar{\epsilon} \psi^i \\ \delta \psi^i &= \epsilon \left(\ \ i\dot{x}^i - \Gamma^i_{jk} \overline{\psi}^j \psi^k \right) \qquad \Rightarrow \ Q = i g_{ab} \overline{\psi}^a \dot{x}^b \ , \ \overline{Q} = Q^\dagger \\ \delta \overline{\psi}^i &= \bar{\epsilon} \left(-i\dot{x}^i - \Gamma^i_{jk} \overline{\psi}^j \psi^k \right) \end{aligned} \tag{7}$$

[2] Superfields are indeed representations of the super-Poincaré algebra in quite the same way that scalar functions and spinors are representations of the Lorentz algebra.

The Noether charges $Q, \overline{Q}$ can be found by the Noether procedure. They generate the transformations for any field ϕ in the sense that $\delta\phi = \left[\epsilon\overline{Q} + \bar{\epsilon}Q, \phi\right]$.

Additionally there is a global $U(1)$ symmetry originating from the R-symmetry group of the super-Poincaré algebra. Namely, in our case the algebra is invariant under a change of $Q \to e^{i\alpha}Q$ and accordingly for $\overline{Q}$ and this carries over to the invariance of the action under

$$\psi \to e^{i\alpha}\psi \ , \ \overline{\psi} \to e^{-i\alpha}\overline{\psi} \quad \Rightarrow \quad F = g_{ab}\overline{\psi}^a\psi^b \ .$$

The Noether charge F is called the fermion number and will give a grading on the Hilbert space that will be constructed momentarily.

2.2. Canonical Quantisation

Now that we've specified the classical theory in consideration, we need to carry out the quantisation of the theory. To that end we will first consider the free theory, i.e. take $h(x) = 0$. Here we will stick to canonical quantisation, i.e. we will promote pairs of canonical variables to operators with canonical (anti-)commutation relations. Namely, working on a local chart (U, x^i) on the target manifold M, define the canonically conjugate momenta for the fields by

$$p_i \equiv \frac{\partial\mathcal{L}}{\partial\dot{x}^i} = g_{ij}\dot{x}^j \quad , \quad \pi_i \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\psi^i)} = ig_{ij}\overline{\psi}^j \quad , \quad \bar{\pi}_i \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\overline{\psi}^i)} = ig_{ij}\psi^j \tag{8}$$

and impose canonical commutation relations to hold on the chart in consideration:

$$\begin{aligned} &\left[x^i, p_j\right] = i\delta^i{}_j & & \longrightarrow \text{Heisenberg algebra} \\ &\left\{\psi^i, \pi_j\right\} = i\delta^i{}_j \quad \Rightarrow \quad \left\{\psi^i, \overline{\psi}^j\right\} = g^{ij} & & \longrightarrow \text{Clifford algebra} \end{aligned}$$

Thus we find that the Hilbert space of the quantum theory should furnish a product representation of these two algebras. As is well known the representations are $L^2(U, \mathbb{C})$ and the exterior algebra in n elements $\bigwedge\{\overline{\psi}^1, \dots, \overline{\psi}^n\}$, respectively. In particular this holds for any local chart U and by gluing over an atlas of the manifold M we conclude that the Hilbert space is given by

$$\mathcal{H} = L^2(M, \mathbb{C}) \times \bigwedge\{\overline{\psi}^1, \dots, \overline{\psi}^n\} \simeq \Omega_{\mathbb{C}}(M) \ .$$

In the last step we identified the exterior algebra of the $\overline{\psi}^i$s with the one of the basis vectors $\mathrm{d}x^i$ of T^*M. The fields act as operators on the Hilbert space by

$$x^i = x^i\cdot \quad , \quad p_i = -i\nabla_i \quad , \quad \overline{\psi}^i = \mathrm{d}x^i\wedge \quad , \quad \psi^i = g^{ij}\,\iota_{\partial_i} \tag{9}$$

where ι_X is the interior product of a differential form with the tangent vector X.

Finally, using (7), the super-Poincaré algebra acts on the Hilbert space $\mathcal{H} = \Omega_{\mathbb{C}}(M)$ as follows:

$$\begin{aligned} Q &= ip_i\overline{\psi}^i = \nabla_i \otimes \overline{\psi}^i\wedge = \partial_i \otimes \mathrm{d}x^i \equiv \mathrm{d} \\ \overline{Q} &= ip_i\psi^i = g^{ij}\partial_i \otimes \iota_{\partial_i} \equiv \mathrm{d}^\dagger \\ H &= \frac{1}{2}\left\{Q, \overline{Q}\right\} = \frac{1}{2}(\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}) = \frac{1}{2}\Delta\ . \end{aligned} \tag{10}$$

Here $\mathrm{d}^{(\dagger)}$ is the (co-)differential and Δ the Laplacian on M. Note that a standard theorem in differential geometry states that for compact M with appropriate boundary conditions[3] the spectrum of the Laplacian has a mass gap.

Accordingly one can check that $F = \overline{\psi}_i\psi^i$ defines a grading on the Hilbert space that coincides with the degree of the differential forms: $F|_{\Omega^p(M)} = p$.

The crucial observation is that we now have exactly the structure specified in theorem 1 and thus can make use of the identification (2). Namely, the supersymmetric ground states of a sigma model are given by the Q-cohomology, which by the identifications (10) coincides with the deRham-cohomology of the target space

$$\mathcal{H}_0 = \bigoplus_{p\in\mathbb{Z}} H^p(Q) = \bigoplus_{p=0}^{n} H^p_{\mathrm{DR}}(M) = H_{\mathrm{DR}}(M)\ .$$

In particular the Witten index is nothing but the Euler characteristic of the target space.

$$\mathrm{Tr}(-1)^F = \sum(-1)^p \dim H^p_{\mathrm{DR}}(M) = \chi(M)$$

2.3. Turning on the potential

The additional terms in the action (6) for $h \neq 0$ imply that the fermions need to transform differently under a SUSY transformation. Indeed working out the details shows that the supercharges will be $Q_h = \overline{\psi}^i(ip_i + \partial_i h)$ and $\overline{Q}_h = {Q_h}^\dagger$, related to our earlier charges by $Q = Q_{h=0} = Q_0$. However it is clear that the canonical conjugate momenta will not change, because the additional terms do not involve temporal derivatives of the fields. Therefore the canonical quantisation is not altered, only the representation of the super-Poincaré generators is modified according to the above changes.

$$\begin{aligned} Q_h &= (\nabla_i + \partial_i h) \otimes \mathrm{d}x^i\wedge = \mathrm{d} + \mathrm{d}h\wedge = e^{-h}\,\mathrm{d}\,e^h \\ \overline{Q}_h &= {\mathrm{d}_h}^\dagger = e^h\,\mathrm{d}^\dagger\,e^{-h} \\ H &= \frac{1}{2}\{\mathrm{d}_h, {\mathrm{d}_h}^\dagger\} = \Delta_h \end{aligned}$$

[3] Trivial: $\partial M = \emptyset$; Dirichlet: $f|_{\partial M} = 0$; Neumann: $\frac{\partial f}{\partial n}|_{\partial M} = 0$, where n is normal to the boundary.

Thus it is the spectrum of the theory changes, as one would expect due to the additional potential energy.

However, the cohomology of $Q_h = e^{-h}Q_0e^h$ is isomorphic to the Q_0-cohomology due to the isomorphism of the complexes:

$$\begin{array}{ccccccccc} 0 & \to & \cdots & \mathcal{H}^p(M) & \xrightarrow{Q_0} & \mathcal{H}^{p+1}(M) & \to & \cdots & \to & 0 \\ & & & \big\downarrow e^h & & \big\downarrow e^h & & & & \\ 0 & \to & \cdots & \mathcal{H}^p(M) & \xrightarrow{Q_h} & \mathcal{H}^{p+1}(M) & \to & \cdots & \to & 0 \end{array}$$

Therefore the ground states of the theory with non-vanishing potential are still given by

$$\mathcal{H}_0 \simeq H_{\mathrm{DR}}(M)$$

and likewise the Witten index doesn't change and only depends on the geometry of the target space $\mathrm{Tr}(-1)^F = \chi(M)$.

3. Landau-Ginzburg Models

A Landau-Ginzburg model (in SQM) is a supersymmetric (1 dimensional) sigma model with *a complex Hermitian target space* (N, h) and a holomorphic superpotential W. In the following we will first discuss the additional structure on N and investigate its implications on the free quantum theory. Afterwards we will turn on the superpotential and state the main differences that arise.

3.1. Complex Hermitian manifolds

A complex manifold N has charts to the open unit disk $D = \{|z| < 1\} \subset \mathbb{C}^n$ and holomorphic transition maps. I.e. locally we have complex coordinates $z^i \in D$ that we can naturally write as $z^i = x^i + iy^i$, where $x^i, y^i \in \mathbb{R}$.

For any complex manifold N this identification of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ gives rise to an integrable almost complex structure J on the tangent space TN with the defining property $J^2 = -\mathrm{Id}$. This allows for a decomposition of the tangent space into the eigenspaces with eigenvalues $\pm i$, respectively spanned by ∂_{z^i} and $\partial_{\bar{z}^{\bar{\imath}}}$. It follows that there is a corresponding decomposition of T^*N and the differential forms.

$$TN_{\mathbb{C}} = TN^+ \oplus TN^- \qquad \to \qquad \Omega^r(N)_{\mathbb{C}} = \bigoplus_{p+q=r} \Omega^{p,q}(N)$$

A Hermitian manifold (N, h) is the complex analogue of a Riemannian manifold, i.e. it is a complex manifold N equipped with a smooth Hermitian product $\langle u, v\rangle = \overline{\langle v, u\rangle}$ on its tangent space, specified by

$$h : TN \times TN \to \mathbb{C}\ ,\ (u, v) \mapsto \langle u, v\rangle = h_{i\bar{\jmath}}\, u^i \bar{v}^{\bar{\jmath}}\ .$$

The complex index notation is simply a mnemonic device so we don't forget to complex conjugate one of the vectors, in particular complex indices are in the same index set $i, \bar{\imath} \in \{1, \ldots, n\}$. Locally the hermitian metric is given by

$$h = h_{i\bar{\jmath}}\, \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{\jmath}} = h_{i\bar{\jmath}} \mathrm{d}z^i \mathrm{d}\bar{z}^{\bar{\jmath}} + h_{i\bar{\jmath}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} \equiv g + i\omega\ ,$$

where g is a Riemannian metric on the underlying real manifold and ω is a symplectic form called the Hermitian form. Note that the Hermitian metric preserves the complex structure $h(u, v) = h(Ju, Jv)$ and one can use this to show $\omega(u, v) = g(Ju, v)$. Thus it is enough to specify a Riemannian metric g or the Hermitian form ω together with the integrable almost complex structure J to fully define a Hermitian manifold (N, h).

A special class of Hermitian manifolds are *Kähler manifolds* for which the Hermitian form ω is closed $\mathrm{d}\omega = 0$ and then called Kähler form. Later it will be crucial that this condition drastically simplifies the Christoffel symbols to $\Gamma^i_{jk} = h^{i\bar{l}} \partial_j h_{k\bar{l}}$.

3.2. Supersymmteric action

In order to build a real-valued action we now need to consider superfields Φ together with their complex conjugates $\bar{\Phi}$. Leaving out the auxiliary field F, the component fields are $(z^i, \psi^i, \overline{\psi}^i)$ and their complex conjugates[4] $(\bar{z}^{\bar{\imath}}, \psi^{\bar{\imath}}, \overline{\psi}^{\bar{\imath}})$.

Going through essentially the same steps as before we find the action in the case of vanishing superpotential $W(z) = 0$.

$$S = \int \mathrm{d}t \left[h_{a\bar{b}}\, \dot{z}^a \dot{\bar{z}}^{\bar{b}} + i h_{a\bar{b}} \left(\overline{\psi}^{\bar{b}}\, \nabla_t \psi^a + \overline{\psi}^a\, \nabla_t \psi^{\bar{b}} \right) + R_{a\bar{b}c\bar{d}}\, \overline{\psi}^a \psi^{\bar{b}}\, \psi^c \overline{\psi}^{\bar{d}} \right]$$

Assuming that we map into a Kähler manifold the action is invariant under the following symmetries:

[4] Due to our earlier choice of putting the fermionic degrees of freedom into a pair of complex conjugates the notation is admittedly a little bulky. We stick to this notation for later convenience, but it's crucial that now ψ^i and $\overline{\psi}^i$ are independent complex degrees of freedom. In particular $\overline{(\psi^i)} = \overline{\psi}^{\bar{\imath}}$ and $\overline{(\overline{\psi}^i)} = \psi^{\bar{\imath}}$, i.e. the complex index notation now also serves to distinguish fields from their complex conjugate.

- $\mathcal{N} = 4$ **extended supersymmetry**, i.e. two independent supersymmetry transformations with complex fermionic parameters $\epsilon_\pm$

$$
\begin{aligned}
&(I) && (II)\\
\delta z^i &= -\epsilon_- \psi^i & \delta z^i &= \epsilon_+ \overline{\psi}^i\\
\delta \psi^i &= i\bar{\epsilon}_- \dot{z}^i & \delta \psi^i &= -\epsilon_+ \Gamma^i_{jk} \overline{\psi}^j \psi^k\\
\delta \overline{\psi}^i &= -\epsilon_- \Gamma^i_{jk} \overline{\psi}^j \psi^k & \delta \overline{\psi}^i &= -i\bar{\epsilon}_+ \dot{z}^i
\end{aligned}
$$

plus corresponding complex conjugate transformations. Note that for a simultaneous transformation with $\epsilon_+ = \bar{\epsilon}_-$ we recover the original $\mathcal{N} = 2$ supersymmetry we observed for a real target manifold. Indeed it is the Kähler condition $d\omega = 0$ which ensures that the decoupled transformations remain a symmetry. The corresponding Noether charges are

$$Q_+ = h_{i\bar{\jmath}} \psi^i \dot{\bar{z}}^{\bar{\jmath}} \ , \quad Q_- = h_{i\bar{\jmath}} \overline{\psi}^i \dot{\bar{z}}^{\bar{\jmath}} \ , \quad + \text{complex conjugates.}$$

- Two global $U(1)$ symmetries $\psi \mapsto \exp(-i\alpha n_\psi)\psi$ with associated Noether charges

n_ψ	ψ^i	$\overline{\psi}^{\bar{\imath}}$	$\overline{\psi}^i$	$\psi^{\bar{\imath}}$		
$U(1)_A$	1	-1	-1	1	$\Rightarrow$	$F_A = h_{i\bar{\jmath}} \left(\overline{\psi}^{\bar{\jmath}} \psi^i - \psi^{\bar{\jmath}} \overline{\psi}^i \right)$
$U(1)_V$	1	-1	1	-1	$\Rightarrow$	$F_V = h_{i\bar{\jmath}} \left(\overline{\psi}^{\bar{\jmath}} \psi^i + \psi^{\bar{\jmath}} \overline{\psi}^i \right)$

F_A is actually the fermion number we found in the real case, where we identified $\overline{\psi}^{\bar{\imath}} = \overline{\psi}^i$ and thus had the same $U(1)$ transformations. However the $U(1)_V$ is new and will give an additional refinement of quantum states.

As before these $U(1)$s really come from the outer automorphism group of the super-Poincaré algebra, which in our current considerations consists of 4 fermionic generators $Q_\pm, \overline{Q}_\pm$ that are determined up to the above phase factors.

3.3. Canonical Quantisation

Following the usual procedure of imposing canonical (anti-)commutation relations, we find two independent copies of Heisenberg and Clifford algebras whose representations are (anti-)holomorphic differential forms, respectively.

$$
\begin{aligned}
&[z^i, p_j] = i\delta^i_j \ , \quad \{\psi^i, \overline{\psi}^{\bar{\imath}}\} = h^{i\bar{\jmath}} && \rightarrow \quad \text{holomorphic}\\
&[\bar{z}^{\bar{\imath}}, p_{\bar{\jmath}}] = i\delta^{\bar{\imath}}_{\bar{\jmath}} \ , \quad \{\psi^{\bar{\imath}}, \overline{\psi}^j\} = h^{\bar{\imath}j} && \rightarrow \quad \text{antiholomorphic}
\end{aligned}
$$

Thus the Hilbert space of the theory is again given by the space of differential forms

$$\mathcal{H} = \Omega(N)_{\mathbb{C}} = \bigoplus_{p,q=0}^{n} \Omega^{p,q}(N)$$

and the fields act on this space analogously to (9). Following the same arguments we thus find that the SUSY generators act as Dolbeault (co-)differentials.

$$\overline{Q}_+ = -i\bar{\partial} \qquad\qquad Q_+ = i\bar{\partial}^\dagger$$
$$Q_- = -i\partial \qquad\qquad \overline{Q}_- = i\partial^\dagger$$

The extended super-Poincaré algebra is

$$\{Q_\alpha, \overline{Q}_\beta\} = H\delta_{\alpha\beta} \ , \ \ \{Q_\alpha, Q_\beta\} = 0 = \{\overline{Q}_\alpha, \overline{Q}_\beta\} \ ,$$

where the Hamiltonian is the Laplacian on the complex manifold $H = \Delta_{\bar{\partial}} = \Delta_\partial = \frac{1}{2}\Delta$.

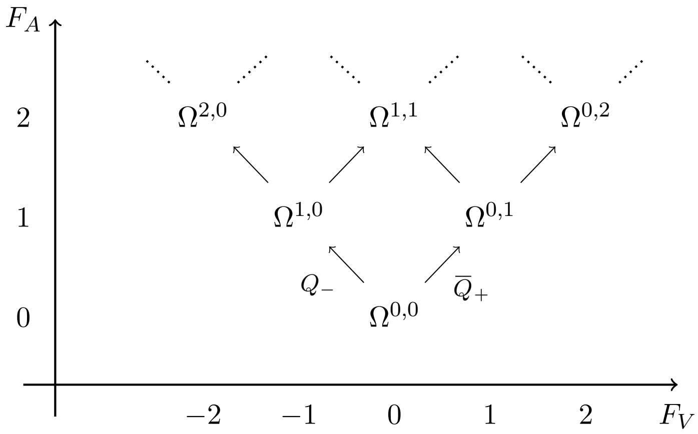
The global $U(1)$ symmetries induce a grading on the Hilbert space $\mathcal{H} = \bigoplus \Omega^{p,q}$:

$$[H, F_{A/V}] = 0 \ , \ \ [F_V, F_A] = 0 \ ; \qquad F_A|_{\Omega^{p,q}(N)} = p + q \ , \ \ F_V|_{\Omega^{p,q}(N)} = -p + q$$

$$\left.\begin{aligned} [F_A, \overline{Q}_\pm] &= \pm\overline{Q}_\pm \\ [F_A, Q_\pm] &= \mp Q_\pm \end{aligned}\right\} \quad \overline{Q}_+,\ Q_- \text{ raise } F_A \text{ by one degree}$$

$$\left.\begin{aligned} [F_V, \overline{Q}_\pm] &= \pm\overline{Q}_\pm \\ [F_V, Q_\pm] &= \pm Q_\pm \end{aligned}\right\} \quad \overline{Q}_+,\ Q_+ \text{ raise } F_V \text{ by one degree}$$

These relations are summarised in the following picture



These considerations show that again theorem 1 holds[5]. Thus the ground states are still given by the deRahm cohomology, but are now further refined by their charge (F_A, F_V), which is related to the Hodge structure induced by the Dolbeault operators via $(F_A, F_V) = (p + q, -p + q)$.

$$\mathcal{H}_0 = \bigoplus_{F_A, F_V = 0}^{n} \mathcal{H}_0^{(F_A, F_V)} = \bigoplus_{p,q=0}^{n} H^{p,q}_{dR}(N) = H_{\mathrm{dR}}(N)$$

[5] Of course under the assumption that the Laplacian has a mass gap. See the corresponding discussion around equation (10).

3.4. Turning on the superpotential

Let $W : N \to \mathbb{C}$ be a holomorphic function; note that then W is only non-trivial, if N is non-compact. Including $W(z = \Phi)$ in the superspace action, it turns out that the additional terms in the Lagrangian are

$$\Delta\mathcal{L} = -\frac{1}{4} h^{i\bar{\jmath}}\, \partial_i W\, \partial_{\bar{\jmath}} \overline{W} - \frac{1}{2}\, \nabla_i \partial_i W\, \psi^i \overline{\psi}^j - \frac{1}{2}\, \nabla_{\bar{\imath}} \partial_{\bar{\jmath}} \overline{W}\, \psi^{\bar{\imath}} \overline{\psi}^{\bar{\jmath}} \ .$$

Extended supersymmetry and $U(1)_A$ symmetry still hold, but $U(1)_V$ symmetry is broken explicitly by the new terms. The supersymmetry transformations will change by terms proportional to ∂W, but these will not affect the canonical momenta and thus this won't change canonical quantisation. Therefore the Hilbert space is given by the differential forms as before, but we will not have the F_V grading on the quantum states.

$$\mathcal{H} = \bigoplus_{l=1}^{2n} \Omega^l(N)$$

The SUSY transformations are altered just like in the real case and since $\mathrm{d}W = (\partial + \bar{\partial})W = \partial W$, the generators are given by

$$\overline{Q}_+ = -i\bar{\partial} - \frac{1}{2}\partial W \wedge \ , \quad Q_- = -i\partial + \frac{1}{2}\partial W \wedge \ , \quad + \text{ conjugate codifferentials} \ .$$

By the usual argument the ground states are given by the Q-cohomology. It actually turns out that the cohomology localises on the middle dimensional holomorphic forms [6]

$$\mathcal{H}_0 = \bigoplus_{l=1}^{2n} \frac{\ker(\overline{Q}_+^l)}{\mathrm{im}(\overline{Q}_+^{l-1})} = \frac{\Omega^{n,0}(N,h)}{\mathrm{d}W \wedge \Omega^{n-1,0}(N,h)} \qquad (11)$$

and thus the Witten index is $\mathrm{Tr}(-1)^{F_A} = \mathrm{Tr}(-1)^n = (-1)^n \dim \mathcal{H}_0$, i.e. all ground states have the same fermion number which only depends on the dimensionality of the target space.

A large class of Landau-Ginzburg models consider the flat target spaces[7] $N = \mathbb{C}^n$. Then (11) simplifies considerably

$$\mathcal{H}_0 = \frac{\Omega^{n,0}(\mathbb{C}^n)}{\partial W \wedge \Omega^{n-1,0}(\mathbb{C}^n)} = \frac{\mathbb{C}[z^1, \ldots, z^n]}{\langle \partial_{z^1} W, \ldots, \partial_{z^n} W \rangle} = \text{Jacobian ring}(W) \ ,$$

where $\mathbb{C}[z^1, \ldots]$ is the ring of polynomials in the coordinate variables and $\langle \partial_{z^1} W, \ldots \rangle$ denotes the ideal generated by the derivatives of the superpotential. The last equality simply is the definition of the Jacobian ring of the function W, which in this context often is called the chiral ring as well. Thus on flat complex target space all information on the ground states is encoded in the critical points of the superpotential $W(z)$.

[6] Equation (11) can be proven by spectral sequence analysis, but is far beyond the scope of these notes. As far as I know the prove makes use of Cartan's theorem B, which holds for Stein manifolds. I do not know if the statement accordingly only holds for Stein manifolds.

[7] Which quite conveniently are Stein.

A. From superspace to a supersymmetric action

In the following the calculation for going from superspace to a more familiar action in terms of bosonic and fermionic fields is presented. We start from the action given in equation (5), namely

$$S = \int_{\mathcal{T}^{1|2}} \left[\Phi^* g - V(\Phi)\right] \ ,$$

and want to reduce this to equation (6). This is done by performing the integration over the fermionic coordinates, i.e. integrate over $\mathrm{d}^2\Theta = \mathrm{d}\Theta\, \mathrm{d}\overline{\Theta}$. That is we simply need to insert the superfield's expansion

$$\Phi(t, \Theta, \overline{\Theta}) = x(t) + \overline{\Theta}\psi(t) + \Theta\overline{\psi}(t) + \overline{\Theta}\Theta F(t)$$

into the action and then extract the parts that are proportional to $\overline{\Theta}\Theta$. We deal with the pullback-metric and the potential term in the action separately, starting with the easier one.

Potential term Since the potential is supposed to be given by a polynomial in Φ and integration is linear, it is sufficient to consider the monomial $V(\Phi) = \Phi^n$. Also we will only consider a one target space coordinate x for pretty much the same reasons. Using the anticommutation properties of the Θs and ψs this expands to

$$\begin{aligned}\Phi^N &= \left(x + \overline{\Theta}\psi + \Theta\overline{\psi} + \overline{\Theta}\Theta F\right)^N = \sum_{k_1+\ldots+k_4=n} \frac{N!}{k_1!\ldots k_4!}\, x^{k_1} \left(\overline{\Theta}\psi\right)^{k_2} \left(\Theta\overline{\psi}\right)^{k_3} \left(\overline{\Theta}\Theta F\right)^{k_4} \\ &= x^N + N\, x^{N-1} \left(\overline{\Theta}\psi + \Theta\overline{\psi}\right) + N(N-1)\, x^{N-2}\, \overline{\Theta}\psi\, \Theta\overline{\psi} + N\, x^{N-1}\, \overline{\Theta}\Theta F \ .\end{aligned}$$

One should pay attention to several minus signs from anticommuting Θs and ψs, but at the end we are left with

$$\int \mathrm{d}^2\Theta\ \Phi^N = N\, x^{N-1} F - N(N-1)\, x^{N-2}\psi\overline{\psi} = \left.\frac{\partial\Phi^N}{\partial x}\right|_{\Theta=\overline{\Theta}=0} F + \left.\frac{\partial^2\Phi^N}{\partial x^2}\right|_{\Theta=\overline{\Theta}=0} \overline{\psi}\psi$$

Now reintroducing multivariable coordinates, non-trivial geometry and an arbitrary polynomial, it is easily seen that the integral of the potential is

$$\int \mathrm{d}^2\Theta\ V\left(\Phi(t, \Theta, \overline{\Theta})\right) = \partial_a V(x)\, F^a + \nabla_a \partial_b V(x)\, \overline{\psi}^a \psi^b \ ,$$

where $V(x) = V\left(\Phi(t, 0, 0)\right)$ and is respectively denoted by $h(x)$ or $W(z)$ in the main parts of the notes.

Pullback metric First we should explain what the pullback of the metric g onto the superspace worldline $\mathcal{T}^{1|2}$ actually is supposed to mean:

$$\Phi^* g = \frac{1}{2} g_{ab}\left(\Phi\right) \overline{D}\Phi^a \, D\Phi^b \; .$$

- The SUSY covariant derivatives are defined by $\{Q, D\} = 0$ for all possible combinations. It can thus be shown that they take the form

$$D = \frac{\partial}{\partial \overline{\Theta}} - i\Theta \frac{\partial}{\partial t} \; , \quad \overline{D} = \frac{\partial}{\partial \Theta} - i\overline{\Theta}\frac{\partial}{\partial t} \; .$$

  These need to be used in the pullback to render the resulting expression supersymmetric. Acting with them on the superfield gives

$$\overline{D}\Phi^a = \overline{\psi}^a - \overline{\Theta}\left(F^a + i\dot{x}^a\right) - i\overline{\Theta}\Theta\dot{\overline{\psi}}^a \tag{12}$$

$$D\Phi^b = \psi^b + \Theta\left(F^b - i\dot{x}^b\right) + i\overline{\Theta}\Theta\dot{\psi}^b \tag{13}$$

- In the above expression we need to evaluate the metric at the point $p \in M$ that the point $(t, \Theta, \overline{\Theta}) \in \mathcal{T}^{1|2}$ is mapped to. To be more precise, we are working on a chart (U, x^i) containing the point $x(t)$. However Φ actually maps adding $\overline{\Theta}\psi^i$ and so on leads to a deviation from this point along the relevant tangent vectors to the point corresponding to p. Expanding in the Θs about $x(t)$ leads to

$$\begin{aligned} g_{ab}\left(\Phi(t,\Theta,\overline{\Theta})\right) =& g_{ab}(x) + \partial_i g_{ab}(x)\left(\overline{\Theta}\psi^i + \Theta\overline{\psi}^i + \overline{\Theta}\Theta F^i\right) \\ &+ \frac{1}{2}\,\partial_i\partial_j g_{ab}(x)\left(\overline{\Theta}\psi^i + \Theta\overline{\psi}^i + \overline{\Theta}\Theta F^i\right)\cdot\left(\overline{\Theta}\psi^j + \Theta\overline{\psi}^j + \overline{\Theta}\Theta F^j\right) \end{aligned}$$

  Multiplying out the last line only leaves one term, so we get

$$g_{ab}(\Phi) = g_{ab}(x) + \partial_i g_{ab}(x)\left(\overline{\Theta}\psi^i + \Theta\overline{\psi}^i + \overline{\Theta}\Theta F^i\right) + \partial_i\partial_j g_{ab}(x)\,\overline{\Theta}\Theta\,\overline{\psi}^i\psi^j \; . \tag{14}$$

The pullback-metric is given by multiplying (12),(13) and (14). Since in the next step we integrate over the Grassmann coordinates, we only need to look out for terms proportional to $\overline{\Theta}\Theta$. These are given by

$$\begin{aligned} \int \mathrm{d}^2\Theta\; \Phi^* g = \;& \frac{1}{2}\,g_{ab} \quad \left(-\dot{x}^a\dot{x}^b - F^aF^b + i\overline{\psi}^a\dot{\psi}^b - i\dot{\overline{\psi}}^a\psi^b\right) \\ +\;& \frac{1}{2}\,\partial_i g_{ab} \quad \left(\psi^i\overline{\psi}^a(F^b - i\dot{x}^b) - \overline{\psi}^i(F^a + i\dot{x}^a)\psi^b + F^i\overline{\psi}^a\psi^b\right) \\ +\;& \frac{1}{2}\partial_i\partial_j g_{ab} \quad \overline{\psi}^i\psi^j\,\overline{\psi}^a\psi^b \end{aligned} \tag{15}$$

The final step will be to integrate out the auxiliary field F: Since there are no time derivatives of F in the action it is not a propagating degree of freedom and we can impose its equation of motion without loosing information on the dynamics of the theory. Note that in doing so we naturally loose off-shell supersymmetry, since then supersymmetry only closes upon use of the equations of motion of the fields. The equation of motion of F is given by the Euler-Lagrange equations

$$0 = \frac{\partial \mathcal{L}}{\partial F^b} = -g_{bi}F^i - \frac{1}{2}\,\partial_i g_{bj}\left(\overline{\psi}^j\psi^i + \overline{\psi}^i\psi^j\right) + \frac{1}{2}\,\partial_b g_{ij}\,\overline{\psi}^i\psi^j + \partial_b V(x)$$

Solving this for F leads to

$$\begin{aligned} F^a &= -\frac{1}{2}g^{ab}\left[\partial_i g_{bj}\overline{\psi}^j\psi^i + \partial_i g_{bj}\overline{\psi}^i\psi^j - \partial_b g_{ij}\overline{\psi}^i\psi^j\right] + g^{ab}\partial_b V(x) \\ &= -\Gamma^a_{ij}\overline{\psi}^j\psi^i + g^{ab}\partial_b V(x)\ . \end{aligned} \tag{16}$$

Plugging this identity into the action leads to the anticipated result. To see this, one simply needs to clean up the mess this produces by reshuffling indices and utilizing that the Riemann tensor in terms of the metric and Christoffel symbols is given by

$$R_{abcd} = \frac{1}{2}\left(\partial_b\partial_c g_{ad} + \partial_a\partial_d g_{bc} - \partial_b\partial_d g_{ac} - \partial_a\partial_c g_{bd}\right) + g_{ij}\left(\Gamma^i_{bc}\Gamma^j{}_{ad} - \Gamma^i{}_{cd}\Gamma^j{}_{ac}\right)\ .$$

Subsequently collecting all terms gives exactly[8] what we want, as one might check explicitly by comparing the following two expressions.

$$\begin{aligned} S = \int \mathrm{d}t\,\Bigg(\ &\frac{1}{2}g_{ab}\ \ \Big(-\dot{x}^a\dot{x}^b + (-\Gamma^a_{ij}\overline{\psi}^j\psi^i + g^{ai}\partial_i V)(-\Gamma^b_{ij}\overline{\psi}^j\psi^i + g^{bi}\partial_i V) \\ &\qquad\qquad + i\overline{\psi}^a\dot{\psi}^b - i\dot{\overline{\psi}}^a\psi^b\ \ \Big) \\ &+ \frac{1}{2}\partial_i g_{ab}\ \ \Big(\ \ \psi^i\overline{\psi}^a\left[(-\Gamma^b_{ij}\overline{\psi}^j\psi^i + g^{bi}\partial_i V) - i\dot{x}^b\right] \\ &\qquad\qquad - \overline{\psi}^i\psi^b\left[(-\Gamma^a_{ij}\overline{\psi}^j\psi^i + g^{ai}\partial_i V) - i\dot{x}^a\right] \\ &\qquad\qquad + (-\Gamma^i_{jk}\overline{\psi}^k\psi^j + g^{ij}\partial_j V)\overline{\psi}^a\psi^b\ \ \Big) \\ &+ \partial_i\partial_j g_{ab}\ \ \overline{\psi}^i\psi^j\,\overline{\psi}^a\psi^b \\ &+ \partial_a V(x)\,(-\Gamma^a_{ij}\overline{\psi}^j\psi^i + g^{ai}\partial_i V) + \nabla_a\partial_b V(x)\,\overline{\psi}^a\psi^b\Bigg) \end{aligned}$$

$$\begin{aligned} = \int \mathrm{d}t\,\Bigg[\ &\overset{(?)}{-}\ \frac{1}{2}\,g_{ab}\,\dot{x}^a\,\dot{x}^b + \frac{i}{2}\,g_{ab}(x)\left(\overline{\psi}^a\,\nabla_t\psi^b - \nabla_t\overline{\psi}^a\,\psi^b\right) - \frac{1}{2}R_{abcd}\,\overline{\psi}^a\psi^b\,\overline{\psi}^c\psi^d \\ &-\frac{1}{2}\,g^{ab}\,\partial_a h(x)\,\partial_b h(x) - \nabla_a\partial_b h(x)\,\overline{\psi}^a\psi^b\Bigg]\ . \end{aligned}$$

where again we defined $\nabla_t\psi^i = \partial_t\psi^i + \Gamma^i_{jk}\,\dot{x}^j\,\psi^k$.

[8]I.e. up to a **minus sign** in front of the x^2-term, which probably comes about by misusage of at least **two different conventions** for factors of i. I wasn't able to nail down, where it breaks down exactly, though.