Heidelberg University

Seminar Vortrag 1 WS2015
SUSY Quantum mechanics

QFT in $dim = 0$

1 Preliminaries on QFT's:

QFT is a theory that aims to bring field theories and quantum mechanics together, under the idea that not only the physical obsevables are quantized but the fields also. We will consider the following basic mathematical concept behind the QFT's :

At first we choose a Manifold $\mathscr{M}$, usualy a Riemannian one with a smooth metric on it.Over this manifold we define fields and our main interesest in QFT is to integrate over the chosen manifold, parametrizing these fields.The last operation is of course our familiar path-integral. An other basic ingriedient of the theory are maps from the base manifold $\mathscr{M}$ to a target manifold $\mathscr{N}$:

$$X : \mathscr{M} \longrightarrow \mathscr{N}$$

Soon we will concider theories of integration in the space of those mappings called *sigma models*.

2 QFT in 0-dim

Today we will take a look at the QFT's for zero dimensional base-manifolds $dim(\mathscr{M}) = 0$. So here we have to deal with:

- Point-like fields: $X : \mathscr{M} \longrightarrow \mathbb{R}$
- An action that is a function of the fields, $S = S[X]$
- and partition functions of the form: $\mathscr{Z} = \int dX e^{-S[X]}$

Example: Let the action integral be

$$S[X] = \frac{\alpha}{2}x^2 + i\epsilon x^3$$

Then:

$$\epsilon = 0 \to \mathscr{Z} = \int dX e^{\frac{(-\alpha)}{2}x^2} = \sqrt{\frac{2\pi}{\alpha}}$$

$$\epsilon \ll 1 \to \mathscr{Z} = \int dX e^{\frac{-(\alpha x^2 - i\epsilon x^3)}{2}} \xrightarrow{pertub.expansion} \int dX \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}x^2} \frac{(-i\epsilon x^3)^n}{n!}$$

At this point is useful to use the method of Feynman diagrams, which will be widely used for partubative computations.

First we consider the function:

Here the physical interpretation of J can be considered to be the source.

$$f(\alpha, J) = \int dX e^{-\frac{\alpha}{2}x^2 + Jx}$$
$$= \int e^{-\frac{\alpha}{2}(x - \frac{J^2}{\alpha})^2 + \frac{J^2}{2\alpha}}$$
$$= \sqrt{\frac{2\pi}{\alpha}} e^{\frac{J^2}{2\alpha}}$$

And we perform the so called *Wick contraction:*
For the function given above, we take the derivatives with respect to J and evaluate them at 0:

$$\frac{\partial^r f}{\partial J^r}\Big|_{\zeta=0} = \int dX X^r e^{-\frac{\alpha}{2}X^2}$$

In order to obtain a non-zero outcome, the partial derivatives $\frac{\partial f}{\partial J}$ must of course show up in pairs, or else the factor $\frac{J}{\alpha}$ coming from the exponential would give 0 after the evaluation. Now every derivative corresponds to an X,so we compute the integral over X^r using the Gaussian measure[1] by considering all possible pairings and contracting them:

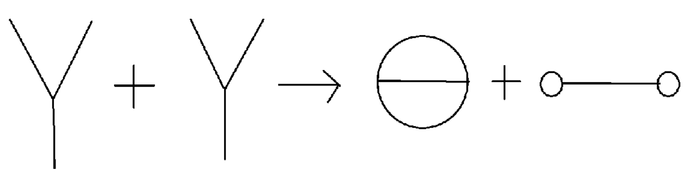
Every pair of $\frac{\partial f}{\partial J}$ gives a factor $\frac{1}{\alpha}$,so $\frac{\partial^r f}{\partial J^r}$ gives

$$(\frac{1}{\alpha})^{\frac{r}{2}} \times (\text{number of all possible contractions})$$

We now use the Wick contraction to calculate the partition function of our example by taking into account the non trivial correction to $\mathscr{Z}(\alpha, 0)$ given by:

$$\mathcal{O}(\epsilon)^2 = \frac{(-i\epsilon)^2}{2!} \int dX X^3 \times X^3 \times e^{-\frac{\alpha}{2}X^2}$$

the graphical representation of which , is the following:



From the two graphs we get as total number of possible pairs of contraction $3! + 3^2 = 15$

$$\implies \int e^{-\frac{\alpha}{2}x^2} \frac{(-i\epsilon x^3)^2}{2} dX = 15 \frac{(-i\epsilon)^2}{2} \sqrt{\frac{2\pi}{\alpha}} \left(\frac{1}{\alpha}\right)^3$$

[1] *Gaussian measure* $\gamma^n : \mathscr{B}(\mathbb{R}^n) \to [0, 1]; \gamma^n(A) = \frac{1}{\sqrt{2\pi}^n} \int_A e^{-\frac{1}{2}\|x\|^2_{\mathbb{R}}} d\lambda^n(x)$

3 Grassmann numbers (fermionic variables)

The Grassmann numbers are elements in an algebra $\mathscr{A}$ over a field $\mathbb{F}$ that contains the reals, such that the following is true: Let $\mathscr{S} = \{\eta_1, ..., \eta_n\}$ be a set of Grassmann numbers, then holds:

$$\{\eta_i, \eta_j\} = \eta_i\eta_j + \eta_j\eta_i = 0 \ \forall i, j$$

and

$$[\eta_i, a] = 0 \ \forall a \in \mathbb{F}$$

Such set of elements in $\mathscr{A}$, with this property of anti-commutativity , spann a Grassmann algebra as a subalgebra of the exterior algebra over $\mathscr{A}$:

$$\mathscr{G} = span_{\mathbb{F}}(\eta_1, ..., \eta_n) < \Lambda(\mathscr{A})$$

So is $\eta_i^2 = 0$ and so $\eta_i^m = 0 \ \forall m \in \mathbb{N}$ (2nd-order nilpotency).This property implies:

$$e^{\eta_i} = \sum_{m=0}^{\infty} \frac{\eta_i^m}{m!} = 1 + \eta_i$$

In generall we see for a function, on Grassmann valued variables or "Grassmann variables" $(\eta_1, ..., \eta_n)$,that its Taylor expansion breaks up after the nth Order.

$$f(\eta_1, ..., \eta_n) = a_0 + \sum_i \eta_i a_i + \frac{1}{2} \sum_{i_1, i_2} \eta_{i_1}\eta_{i_2} a_{i_1,i_2} + ... + \frac{1}{n!} \sum_{i_1,...,i_n} \eta_{i_1} \cdots \eta_{i_n} a_{i_1,...,i_n}$$

3.1 Differentiation of Grassmann variables

For the Grassmann variables $(\eta_1, ..., \eta_n)$ we have:

$$\frac{\partial \eta_i}{\partial \eta_j} = \delta_{ij}, \frac{\partial \eta_j \eta_i}{\partial \eta_i} = \frac{\partial(-\eta_i\eta_j)}{\partial \eta_i} = -\eta_j$$

So it is from the Taylor expansion:

$$\frac{\partial}{\partial \eta_n} \cdots \frac{\partial}{\partial \eta_1} f(\eta_1, ..., \eta_n) = a_{1,...,n}$$

The use of Grassman variables "Fermionic variables" will lead us to the path integral formulation for fermionic fields.

3.2 Integration of Grassmann variables (Berezin integral)

For the Integration of Grassmann variables the following hold:

- $\int d\eta \ cf(\eta) = c \int d\eta f(\eta)$ (linearity)
- $\int d\eta \ f(\eta) = \int d\eta f(\eta + a)$ (invariance under transformations)

From the last property one can follow that for a function g in one Grassmann variable η with the property $g(\eta) = \eta$ it is:

$$\int d\eta \ \eta = \int d\eta(\eta + a) \implies \int d\eta a = 0$$

Further we choose $\int d\eta\eta = 1$, then it is for the function $f(\eta_1, ..., \eta_n)$:

$$\int d\eta_n \cdots d\eta_1 f(\eta_1, ..., \eta_n) = a_{1,...,n}$$

If we go back to that we have obtained by the differentiation of the same function, we see that integration and differentiation are giving the same result.This will help us to calculate partition functions of bosonic and fermionic variables.

Example: Let $S(X, \Psi_1, \Psi_2) = S_0 - \Psi_1\Psi_2 S_1(X)$ be an action with X bosonic and $\Psi_1; \Psi_2$ fermionic variables. Then is the partition function over this action:

$$\begin{aligned}
\mathscr{Z} &= \int dX d\Psi_1 d\Psi_2 e^{-S_0 + \Psi_1\Psi_2 S_1(x)} \\
&= \int dX d\Psi_1 d\Psi_2 e^{-S_0}(1 + \Psi_1\Psi_2 S_1(X)) \\
&= \underbrace{\int dX d\Psi_1 d\Psi_2 e^{-S_0}}_{\text{Grassman int.}} + \int dX d\Psi_1 d\Psi_2 e^{-S_0(X)} S_1(X) \\
\implies \mathscr{Z} &= \int dX e^{-S_0(X)} S_1(X)
\end{aligned}$$

4 SUSY

For a special choice of S_0, S_1 as : $S_0(X) = \frac{1}{2}(\partial h)^2; S_1(X) = \partial^2 h$ for a real function h,the system has a symmetry which is obtained as a relation between the bosonic and fermionic fields, called supersymmetry (SUSY).This symmetry is given by the following field-transformations:

$$\begin{aligned}
\delta_\epsilon \epsilon_1 \ &= \Psi_1 + \epsilon\Psi_2 \\
\delta\Psi_1 &= \epsilon_2 \partial h \\
\delta\Psi_2 &= -\epsilon_1 \partial h
\end{aligned}$$

for $\epsilon_i; \Psi_i$ fermionic.
Under the above transformation the defining action of the QFT:

$$S(X, \Psi_1, \Psi_2) := \frac{1}{2}(\partial h)^2 - \partial^2 h \Psi_1 \Psi_2$$

stays invariant:

$$
\begin{aligned}
S \longrightarrow & S + \frac{\partial S}{\partial X}\delta X + \frac{\partial S}{\partial \Psi_1}\delta\Psi_1 + \frac{\partial S}{\partial \Psi_2}\delta\Psi_2 \\
&= S + h'h''(\epsilon_1\Psi_1 + \epsilon_2\Psi_2) + h''(\epsilon_1\Psi_1 + \epsilon_2\Psi_2)\Psi_1\Psi_2 + h''\Psi_2\epsilon_2 h' + h''(-\Psi_1)(-\epsilon_1 h') \\
&= S + h'h''(\epsilon_1\Psi_1 + \epsilon_2\Psi_2) - h'h''(\epsilon_1\Psi_1 + \epsilon_2\Psi_2) \\
&= S
\end{aligned}
$$

5 Localization principle in SUSY

5.1 The idea of the localization principle

The principle is a way to calculate integrals with respect to both bosonic and fermionic variables. If for our system we have a given supersymmetric transformation then the integral becomes localized in the field configurations for which the fermionic variables are invariant under the supersymmetry. We use it to compute partition functions by reducing the dimension of the path integrals defining the QFT. (For further reading in a more general framework:[2])

Now we want to discuss the use of the localization principle in the supersymmetric context as we have seen it in the last paragraph (Prg.4). In this context the statement can be exact formulated as :

"The path integral is localized at loci where the right hand side of the fermionic transformation under supersymmetry is 0", that is $\partial h = 0$.

To explain this we start with the contraposition by considering h such that $\partial h \neq 0$ and we imply that then the partion function $\mathscr{Z}$ will vanish. Let the partition function be:

$$\mathscr{Z} := \int e^{-S} dX d\Psi_1 \Psi_2$$

To show that $\mathscr{Z} = 0$ we perform a SUSY-transformation to set one of the fermions in the action to be 0 and use the rules of Grassmann integration. We change the bosonic variable into:

$$X := X' + \frac{\Psi_1\Psi_2}{\partial h(X)}, \text{where } \partial h \neq 0 \text{ by assumption}$$

And by using the invariance of the action under SUSY we get:

$$S(X, \Psi_1, \Psi_2) = S(X', 0, \Psi_2') = S(X')$$

and for:

$$X = X' + g(X')\Psi_1\Psi_2, \text{ with } g(X') = \frac{1}{\partial h(X')}$$

[2]

Supersymmetry and localization: `http://arxiv.org/abs/hep-th/9511112`
Or take a look at Adam's lecture Week 12 on "Equivariant localization".

then this implies:

$$\begin{aligned}
\Longrightarrow \mathscr{Z} &= \frac{1}{\sqrt{2\pi}} \int dX' d\Psi_1 d\Psi_2 e^{-S(X')} \frac{dX}{dX'} \\
&= \frac{1}{\sqrt{2\pi}} \int dX' d\Psi_1 d\Psi_2 e^{-S(X')} (1 + \partial g(X')\Psi_1\Psi_2) \\
&= \frac{1}{\sqrt{2\pi}} \int dX' e^{-S(X')} \partial g(X') \\
&= \frac{1}{\sqrt{2\pi}} \int d[g(X')e^{-S(X')}] = 0
\end{aligned}$$

because $\delta S(X') = 0$.

So we examine now how the system behaves near the critical points of the function h. We study this for the case where h is a generic polynomial. So $h \in \mathbb{C}[X]$, i.e. h has finitely many critical points x_c:

$$h = h(x_c) + \frac{h''(x_c)}{2}(x - x_c)^2 + \cdots, \text{and set } \alpha_c := h''(x_c)$$

Then the partition function takes the form:

$$\begin{aligned}
\mathscr{Z} &= \int \frac{dX d\Psi_1 \Psi_2}{\sqrt{2\pi}} e^{\frac{-\alpha_c^2}{2}(x-x_c)^2 + \alpha_c \Psi_1 \Psi_2} = \int \frac{dX}{\sqrt{2\pi}} \alpha_c e^{\frac{-\alpha_c^2}{2}(x-x_c)^2} \\
&= \sqrt{\frac{2\pi}{\alpha_c^2}} \frac{\alpha_c}{\sqrt{2\pi}} = \frac{\alpha_c}{|\alpha_c|} = sgn(h''(x_c)) \\
&\longrightarrow \mathscr{Z} \in \{0, \pm 1\} (\text{depending on } n := grad(h))
\end{aligned}$$

We see here that the partition function is only effected by $grad(h)$. Surprisingly $\mathscr{Z}$ turns out to be an integer, something that gives away the nature of the partition function as a "counting function".

6 Deformation theory

The idea of deformation theory is to simplify the given problem by changing the function h by applying affine transformations on it,such that the partition function stays invariant under the change of the function h.

$$\text{Let } h \longrightarrow h + \rho, \rho, h \in \mathbb{C}[X] : grad(\rho) < grad(h)$$

(Homothetic affine transformation of trivial ratio in $\mathbb{C}[X]$)

and $f = \delta g, \delta e^{-S} = 0$,where the latest has been implied by the postulate of invariance of the partition function. For this we get:

$$< f >= \int e^{-S} f = \int e^{-S} \delta g = \int \delta(g e^{-S}) = 0$$

and for $g = \partial\rho\Psi_1$

$$
\begin{aligned}
f = \delta g = \partial^2\rho\delta X\Psi_1 + \partial\rho\delta\Psi_1 \\
= \epsilon(-\partial^2\rho\Psi_1\Psi_2 + \partial\rho\partial h) \\
\longrightarrow < \partial\rho\partial h - \partial^2\rho\Psi_1\Psi_2 >= 0 = \delta S
\end{aligned}
$$

@BookMirror Symmetry, author = K.Hori, S.Katz, A.Klemm,R.Pandharipande,R.Thomas, C.Vafa, R.Vakil, E.Zaslow,