Gauged Linear Sigma Models
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1 Aim and Basic Results

• Up to now: Landau-Ginzburg (LG) and Sigma Models on Calabi-Yau (CY) are described differently.

• Witten (1993) proposed a generalization of LG and CY-models to a common, unified model, the Gauged Linear Sigma Model (GLSM). Schematically:

![Diagram showing Calabi-Yau and Landau-Ginzburg correspondence](image)

Figure 1: Calabi Yau/Landau-Ginzburg correspondence, from [1].

$\rightarrow$ CY- and LG-models can be interpreted as different phases of the same system.

• $r$ plays the role of a tuning parameter for the ’phase transition’.
• Also remarkable: The elliptic genus of an $\mathcal{N} = (2, 2)$ supersymmetric theory, defined as

$$Z_{T^2}(\tau, z, u) = \text{Tr}_{RR}(-1)^F q^{H_I} \bar{q}^{H_R} y^J \prod_a x_a^K_a$$  \hspace{1cm} (1.1)

is a topological invariant which may be calculated for the GLSM using modern techniques. (See [1].)

2 Supersymmetric gauge theories

2.1 Revision: Gauge-invariance in scalar field theory

Aim: We first need to define the Lagrangian of a supersymmetric gauge theory.

To this end, we review the standard procedure of introducing a gauge field into a $U(1)$-symmetric scalar field theory with Lagrangian

$$\mathcal{L} = -\sum_{i=1}^{n} |\partial_\mu \phi^i|^2 - U(\phi)$$ \hspace{1cm} (2.1)

where

$$U(\phi) = \frac{e^2}{2} \left( \sum_{i=1}^{n} |\phi^i|^2 - r \right)^2.$$ \hspace{1cm} (2.2)

For the sake of completeness, we introduce the vacuum manifold $M_{\text{vac}}$:

**Definition 1.** The set of classical vacua $M_{\text{vac}}$ is defined as the set of all configurations $\phi = (\phi^1, ..., \phi^n)$ where $U(\phi)$ attains its minimum value, i.e.

$$M_{\text{vac}} = \{ \phi = (\phi^1, ..., \phi^n) \in \mathbb{C}^n : U(\phi) = 0 \}. \hspace{1cm} (2.3)$$

Note that for $r < 0$, $M_{\text{vac}} = \{0\}$ consists of a single point, while for $r > 0$, $M_{\text{vac}} = S^{n-1}_{\sqrt{r}}$ is a sphere of radius $\sqrt{r}$. One could now go in detail about this so-called spontaneous symmetry breaking for $r > 0$, but we will not do so here. However, a similar argument involving the structure of $M_{\text{vac}}$ will appear when we discuss the different phases of the GLSM.

The Lagrangian (2.1) is invariant under the global $U(1)$-transformation

$$(\phi^1(x), ..., \phi^n(x)) \longrightarrow (e^{i\gamma} \phi^1(x), ..., e^{i\gamma} \phi^n(x)),$$ \hspace{1cm} (2.4)

which is to be understood as a global phase rotation, where $\gamma \in \mathbb{R}$ is a real number.

This is however not true anymore, if $\gamma$ is allowed to depend on the space-time coordinates $\gamma \equiv \gamma(x)$, since

$$\partial_\mu \phi^i(x) \longrightarrow \partial_\mu \left( e^{i\gamma(x)} \phi^i(x) \right) = e^{i\gamma(x)} (\partial_\mu + i \partial_\mu \gamma(x)) \phi^i(x).$$ \hspace{1cm} (2.5)

The invariance can be restored by introducing a vector field (or: one-form field) $v_\mu(x)$ as an additive contribution to the partial derivative. This gives the following
**Definition / Lemma 2.** The covariant derivative is defined as

\[ D_\mu \varphi^j(x) = (\partial_\mu + iv_\mu(x)) \varphi^j(x). \]  

(2.6)

The Lagrangian

\[ \mathcal{L} = - \sum_{i=1}^n |D_\mu \varphi^i|^2 - U(\varphi) \]  

(2.7)

is invariant under the combined gauge transformation

\[
\begin{align*}
\varphi^i(x) &\rightarrow e^{i\gamma(x)}\varphi^i(x) \\
v_\mu(x) &\rightarrow v_\mu(x) - \partial_\mu \gamma(x).
\end{align*}
\]  

(2.8)

Notice: \( \mathcal{L} \equiv \mathcal{L}_{\text{kin}} \) defined in (2.7) contains:

- a kinetic term for the \( \varphi \) fields,
- interaction terms between the \( v_\mu \) and the \( \varphi \) fields,

but no kinetic term for the \( v_\mu \). Therefore, one could consider \( v_\mu \) as an auxiliary field and eliminate it using its equations of motion. If \( v_\mu \) is to be considered as a physical field, such as the photon in ordinary QED, we need to add a kinetic term for it into the Lagrangian.

Indeed, the definition

\[ \mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} \]  

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{2e^2} v_{\mu\nu} v^{\mu\nu} \]  

(2.9)

with \( v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \) (the curvature or field strength of the gauge field) gives a gauge-invariant theory where \( v_\mu \) has a kinetic term.

### 2.2 Gauge-invariance in supersymmetric QFT

Now we want to mimic this procedure for a superfield \( \Phi \) instead of a scalar field \( \varphi \). Recall (i.e. from Talk 3 or chapter 12 of [2]) that a \( \mathcal{N} = (2,2) \) chiral superfield in 2 dimensions (which will be our main interest here) has coordinates \( x^0, x^1, \theta^\pm, \bar{\theta}^\pm \).

The coordinates \( \theta^\pm \) and \( \bar{\theta}^\pm \) are anticommuting, and hence fulfill \( (\theta^\pm)^2 = 0 = (\bar{\theta}^{\pm})^2 \), so employing a Taylor-expansion-like argument, one can see that \( \Phi \) can be expanded as

\[
\Phi(x^\mu, \theta^\pm, \bar{\theta}^{\pm}) = \varphi - i\theta^+ \bar{\theta}^+ \partial_+ \varphi - i\theta^- \bar{\theta}^- \partial_- \varphi - \theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- \partial_+ \partial_- \varphi + \theta^+ \psi_+ - i\theta^+ \theta^- \bar{\theta}^- \partial_+ \psi_+ + \theta^- \psi_- - i\theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^- \partial_- \psi_- + \theta^+ \theta^- F. \]  

(2.10)

Here \( x^\pm = x^0 \pm x^1 \) and \( \partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2} (\partial_0 \pm \partial_1) \) are the derivatives with respect to these coordinates. The fields \( \varphi, F \) and \( \psi_\pm \) are fields in ordinary space, i.e. functions of \( x^\mu \) only.
The equivalent of the theory (2.1) (without potential), is the \textit{manifestly SUSY invariant} Lagrangian $\mathcal{L}$:

$$\mathcal{L} = \int d^4\theta \bar{\Phi} \Phi = \int d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \bar{\Phi} \Phi. \quad (2.11)$$

The integration with respect to $d^4\theta$ extracts the component of $\bar{\Phi} \Phi$ that contains terms proportional to $\theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^-$ with appropriate sign.

The Lagrangian (2.11) was chosen such that it admits just as its scalar field theoretic counterpart (2.1) — a global phase rotation-invariance, that is, $\mathcal{L}$ is unchanged under

$$\Phi \rightarrow e^{i \alpha} \Phi. \quad (2.12)$$

Now replace $\alpha$ by a \textit{chiral superfield} $A \equiv A(x, \theta^\pm, \bar{\theta}^\mp)$. Again, since one has $\Phi \rightarrow e^{iA} \Phi$, the Lagrangian is not invariant under this local transformation anymore, since

$$\bar{\Phi} \Phi \rightarrow \bar{\Phi} e^{-iA+iA} \Phi. \quad (2.13)$$

The way out is again the introduction of an auxiliary field $V$, which is another chiral superfield with appropriate transformation behaviour. This leads to

\textbf{Lemma 3. For a chiral superfield $V$ with transformation behaviour}

$$V \rightarrow V + i(\bar{A} - A), \quad (2.14)$$

\textit{the Lagrangian}

$$\mathcal{L}_{\text{kin}} = \int d^4\theta \bar{\Phi} e^V \Phi \quad (2.15)$$

is invariant under the combined gauge transformation

\begin{align*}
\left\{ \begin{array}{l}
\Phi(x, \theta^\pm, \bar{\theta}^\pm) \rightarrow e^{iA(x, \theta^\pm, \bar{\theta}^\pm)} \Phi(x, \theta^\pm, \bar{\theta}^\pm) \\
V(x, \theta^\pm, \bar{\theta}^\pm) \rightarrow V(x, \theta^\pm, \bar{\theta}^\pm) + i(\bar{A}(x, \theta^\pm, \bar{\theta}^\pm) - A(x, \theta^\pm, \bar{\theta}^\pm)).
\end{array} \right. 
\end{align*} \quad (2.16)

A real superfield with transformation behaviour (2.14) is called a \textit{vector superfield}. One can use the gauge invariance of $V$ to bring its expansion in terms of the $\theta^\pm, \bar{\theta}^\mp$ into the form

$$V = \theta^- \bar{\theta}^- (v_0 - v_1) + \theta^+ \bar{\theta}^+ (v_0 + v_1) - \theta^- \bar{\theta}^+ \sigma - \theta^+ \bar{\theta}^- \bar{\sigma} + i\theta^- \theta^+(\bar{\sigma}^- \lambda_- + \bar{\sigma}^+ \lambda_+) + i\bar{\theta}^+ \bar{\theta}^- (\theta^- \lambda_- + \theta^+ \lambda_+) + \theta^- \theta^+ \bar{\theta}^- \bar{\theta}^+. \quad (2.17)$$

In this expansion, the fields have the following statistics:

- \textbullet{} $\lambda_\pm, \bar{\lambda}_\pm$ define a \textit{Dirac fermion field}
- \textbullet{} $D$ defines a \textit{real scalar field}
- \textbullet{} $\sigma, \bar{\sigma}$ define a \textit{complex scalar field} and
- \textbullet{} $v_0, v_1$ define a \textit{one-form field}.

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The gauge where $V$ can be written in the above form is referred to as Wess-Zumino gauge.

Notice that there is still a residual gauge symmetry, i.e. gauge transformations that keep the form (2.17), and it is given by

$$v_\mu(x) \rightarrow v_\mu(x) - \partial_\mu \alpha(x),$$

with all other component fields unchanged. Two natural questions arise:

1. How can $v_{\mu\nu}$ be generalized to a supersymmetric field strength?
2. Which of the various fields in the vector multiplet $V$ get kinetic terms, and which are to be eliminated using the equations of motion?

**Definition 4.** For a vector multiplet $V$, the super-field strength is defined as

$$\Sigma = D_+ D_- V.$$  \hspace{1cm} (2.19)

Just like $v_{\mu\nu}$, $\Sigma$ is invariant under the gauge transformation (2.14). The kinetic term for $V$ is given in terms of $\Sigma$ as

$$L_{\text{gauge}} = -\frac{1}{2e^2} \int d^4 \theta \Sigma \Sigma.$$  \hspace{1cm} (2.20)

By a straightforward, but tedious calculation, one can obtain the component expansions of $L_{\text{kin}}$ and $L_{\text{gauge}}$:

$$L_{\text{gauge}} = \frac{1}{2e^2} (\ldots)$$

$$L_{\text{kin}} = \ldots$$

One can also write down so called twisted $F$-terms for $\Sigma$. The most important choice for us is

$$\tilde{W}_{FI, \vartheta} = -t \Sigma = -r \Sigma + i \vartheta \Sigma,$$  \hspace{1cm} (2.23)

with $t = r - i \vartheta$, $r$ being called Fayet-Iliopoulos parameter\(^1\) and $\vartheta$ the theta angle. The corresponding contribution to the Lagrangian is

$$L_{FI, \vartheta} = \frac{1}{2} \left(-t \int d^2 \tilde{\theta} \Sigma + \text{c.c.}\right) = -r D + \vartheta v_{01}.$$  \hspace{1cm} (2.24)

Here, the integration is defined as $d^2 \tilde{\theta} = d\vartheta^- d\vartheta^+$. The final result is the Lagrangian for the Gauged Linear Sigma Model

$$L = L_{\text{kin}} + L_{\text{gauge}} + L_{FI, \vartheta} + L_W = \int d^4 \theta \left( \Phi V \Phi - \frac{1}{2e^2} \Sigma \Sigma \right) + \frac{1}{2} \left(-t \int d^2 \tilde{\theta} \Sigma + \text{c.c.}\right) + L_W.$$  \hspace{1cm} (2.25)

\(^1\)hence the subscripts 'FI'.
Comments:

• $L_W$ is a Lagrangian contribution from a superpotential (yet to be introduced).

• $D, F$ have no kinetic term and can be eliminated using the equations of motion.

• After said elimination, one can extract the following potential energy term for $\sigma, \varphi$ (neglecting the superpotential):

$$U = |\sigma|^2|\varphi|^2 + \frac{e^2}{2} (|\varphi|^2 - r)^2. \quad (2.26)$$

3 The different phases of the model

3.1 $\mathbb{CP}^{N-1}$ sigma model (no superpotential)

Consider a $U(1)$ gauge theory with $N$ chiral superfields $\Phi_1, ..., \Phi_N$:

$$\mathcal{L} = \int d^4\theta \left( \sum_{i=1}^{N} \Phi_i e^V \Phi_i - \frac{1}{2e^2} \Sigma \Sigma \right) + \frac{1}{2} \left( -t \int d^2\tilde{\theta}\Sigma + c.c. \right). \quad (3.1)$$

After eliminating $D$ and $F_i$, one obtains again a potential energy term for $\sigma$ and $\varphi_i$:

$$U = \sum_{i=1}^{N} |\sigma|^2|\varphi_i|^2 + \frac{e^2}{2} \left( \sum_{i=1}^{N} |\varphi_i|^2 - r \right)^2. \quad (3.2)$$

From this, one can discuss where $U$ attains 0 for different values of the (real) Fayet-Iliopoulos parameter $r$:

• $r > 0$: $U = 0$ can only be attained if $\sum_{i=1}^{N} |\varphi_i|^2 = r > 0$. Then, $\exists 1 \leq i \leq N$: $|\varphi_i|^2 > 0$, so $\sigma = 0$.

• $r = 0$: $U = 0$ attained if $\varphi = 0$, $\sigma$ arbitrary.

• $r < 0$: $U > 0$ for all configurations, so there is no zero energy ground state.

For $r > 0$: The set of all classical vacua modulo the $U(1)$ gauge group forms the vacuum manifold, since we require configurations that can be transformed into one another by gauge transformations to be physically equivalent.

In our case, this is

$$\left\{ (\varphi_1, ..., \varphi_N) \middle| \sum_{i=1}^{N} |\varphi_i|^2 = r \right\} / U(1) = \mathbb{CP}^{N-1} \quad (3.3)$$

An analysis of the excitations from the vacuum manifold reveals that in this model, the gauge fields $v_\mu$ acquire mass due to the superHiggs mechanism, which can be thought of as the supersymmetric generalization of the Higgs mechanism (see [2]).
3.2 Hypersurfaces in $\mathbb{CP}^{N-1}$

Consider a polynomial $G$ of degree $d$ in the variables $\varphi_1, ..., \varphi_N$:

$$G(\varphi_1, ..., \varphi_N) = \sum_{i_1, i_2, ..., i_d} a_{i_1, i_2, ..., i_d} \varphi_{i_1} \cdot ... \cdot \varphi_{i_d}. \quad (3.4)$$

**Definition 5.** A polynomial (3.4) is called generic or transverse if the following implication holds:

$$G(\varphi) = \frac{\partial G}{\partial \varphi_1}(\varphi) = ... = \frac{\partial G}{\partial \varphi_N}(\varphi) = 0 \implies \varphi_1 = ... = \varphi_N = 0. \quad (3.5)$$

The polynomial $G$ defines the hypersurface $M$ of $\mathbb{CP}^{N-1}$ as

$$M = \{ \varphi \in \mathbb{CP}^{N-1} | G(\varphi_1, ..., \varphi_N) = 0 \}. \quad (3.6)$$

$M$ is a smooth complex manifold with (complex) dimension $N - 2$, which justifies its interpretation as a hypersurface.

Consider a $U(1)$ gauge theory with $N + 1$ chiral multiplets $\Phi_1, ..., \Phi_N, P$ such that:

$$\begin{cases} 
\Phi_1, ..., \Phi_N \to U(1)\text{-charge } 1, \\
P \to U(1)\text{-charge } -d.
\end{cases} \quad (3.7)$$

Then the GLSM Lagrangian with superpotential

$$W = P \cdot G(\Phi_1, ..., \Phi_N) \quad (3.8)$$

is given by

$$\mathcal{L} = \int d^4\theta \left( \sum_{i=1}^{N} \Phi_i e^V \Phi_i + \overline{P} e^{-dV} P - \frac{1}{2e^2} \sum \overline{\Sigma} \right) - \frac{1}{2} \left( \int d^2\tilde{\overline{\Sigma}} + \text{c.c.} \right) - \frac{1}{2} \left( \int d^2\theta P \cdot G(\Phi_1, ..., \Phi_N) + \text{c.c.} \right). \quad (3.9)$$

From this, one can extract the potential term for the scalar fields as

$$U = |\sigma|^2 \sum_{i=1}^{N} |\varphi_i|^2 + |\sigma|^2 d^2|p|^2 + \frac{e^2}{2} \left( \sum_{i=1}^{N} |\varphi_i|^2 - d|p|^2 - r \right)^2$$

$$+ \frac{1}{4} |G(\varphi_1, ..., \varphi_N)|^2 + \frac{1}{4} \sum_{i=1}^{N} |p|^2 |\partial_i G|^2. \quad (3.10)$$

Here, $p$ means the scalar field component of $P$. The sign of $r$ will determine the structure of the vacuum manifold.

The main analysis will now be to set the right-hand side of equation (3.10) to zero and determine the configurations for $\sigma, p$ and $\varphi_i$ that fulfill the equation.
3.2.1 $r > 0$: Calabi-Yau regime

If $r > 0$, $U = 0$ requires $\varphi_i \neq 0$ for one $i$, therefore $\sigma = 0$.
Assume $p \neq 0$, then $G = \partial_1 G = \ldots = \partial_N G = 0$. By transversality, one has
\[
\varphi_1 = \ldots = \varphi_N = 0,
\] (3.11)
which is a contradiction. It follows that $p = 0$. To sum everything up, $U = 0$ is attained if and only if
\[
p = \sigma = 0 \quad \text{and} \quad \sum_{i=1}^{N} |\varphi_i|^2 = r \quad \text{and} \quad G(\varphi_1, \ldots, \varphi_N) = 0.
\] (3.12)
The vacuum manifold is now the set of all fields satisfying the above equations modulo $U(1)$. This is indeed the hypersurface $M \subseteq \mathbb{CP}^{N-1}$. One can now show, that the requirement $d = N$ makes $M$ a so called Calabi-Yau manifold.

**Definition 6.** A Calabi-Yau manifold is a compact Kähler manifold with vanishing first chern class.

**Theorem 7.** A smooth hypersurface $M \subseteq \mathbb{CP}^{N-1}$ of degree $d$ is a Calabi-Yau manifold if and only if $d = N$.\(^2\)

The last statement shows that for $r > 0$, the GLSM reduces to a non-linear sigma model on a Calabi-Yau manifold if $N = d$.

3.2.2 $r < 0$: Landau-Ginzburg regime

If $r < 0$, $U = 0$ requires $p \neq 0$, therefore again $\sigma = 0$.
Since $G = \partial_1 G = \ldots = \partial_N G = 0$, transversality implies $\varphi = 0$, therefore
\[
|p| = \sqrt{\frac{|r|}{d}}
\] (3.13)
Any choice of the vacuum, i.e. $\langle p \rangle = \sqrt{|r|/d}$ breaks the gauge invariance, and by the super-Higgs mechanism, the vector multiplets and the $P$-multiplets gain mass $e^{\sqrt{|r|/d}}$.

If one takes $e \to \infty$, the massive modes decouple from the classical theory and we are left with a theory of the $\Phi_i$ fields only, which is a Landau-Ginzburg theory with superpotential
\[
W = \langle p \rangle G(\Phi_1, \ldots, \Phi_N).
\] (3.14)

References


\(^2\)The reason is that the first chern class of $M$ is given as $c_1(M) = (N - d)[H]|_M$. See [2].