Gauged Linear Sigma Models

Seminar – Supersymmetry in Geometry and Quantum Physics, WS 2015/16

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Contents

1 Aim and Basic Results

- Up to now: *Landau-Ginzburg (LG)* and Sigma Models on *Calabi-Yau (CY)* are described differently.
- Witten (1993) proposed a generalization of LG and CY-models to a common, unified model, the Gauged Linear Sigma Model (GLSM).
  Schematically:

Figure 1: Calabi Yau/Landau-Ginzburg correspondence, from [1].

$\longrightarrow$ CY- and LG-models can be interpreted as different *phases* of the same system.

- r plays the role of a tuning parameter for the 'phase transition'.

- Also remarkable: The *elliptic genus* of an $\mathcal{N}=(2,2)$ supersymmetric theory, defined as

$$Z_{T^2}(\tau,z,u)=\mathrm{Tr}_{\mathrm{RR}}(-1)^F q^{H_L}\bar{q}^{H_R}y^J\prod_a x_a^{K_a} \tag{1.1}$$

is a topological invariant which may be calculated for the GLSM using modern techniques. (See [1].)

2 Supersymmetric gauge theories

2.1 Revision: Gauge-invariance in scalar field theory

Aim: We first need to define the Lagrangian of a supersymmetric gauge theory.

To this end, we review the standard procedure of introducing a gauge field into a $U(1)$-symmetric scalar field theory with Lagrangian

$$\mathcal{L}=-\sum_{i=1}^{n}|\partial_\mu\varphi^i|^2-U(\varphi) \tag{2.1}$$

where

$$U(\varphi)=\frac{e^2}{2}\left(\sum_{i=1}^{n}|\varphi^i|^2-r\right)^2. \tag{2.2}$$

For the sake of completeness, we introduce the vacuum manifold M_{vac}:

Definition 1. *The set of classical vacua M_{vac} is defined as the set of all configurations $\varphi=(\varphi^1,...,\varphi^n)$ where $U(\varphi)$ attains its minimum value, i.e.*

$$M_{vac}=\{\varphi=(\varphi^1,...,\varphi^n)\in\mathbb{C}^n:U(\varphi)=0\}. \tag{2.3}$$

Note that for $r<0$, $M_{vac}=\{0\}$ consists of a single point, while for $r>0$, $M_{vac}=\mathbb{S}^{n-1}_{\sqrt{r}}$ is a sphere of radius $\sqrt{r}$. One could now go in detail about this so-called *spontaneous symmetry breaking* for $r>0$, but we will not do so here. However, a similar argument involving the structure of M_{vac} will appear when we discuss the different phases of the GLSM.

The Lagrangian (2.1) is invariant under the global $U(1)$-transformation

$$(\varphi^1(x),...,\varphi^n(x))\longrightarrow(e^{i\gamma}\varphi^1(x),...,e^{i\gamma}\varphi^n(x)), \tag{2.4}$$

which is to be understood as a global phase rotation, where $\gamma\in\mathbb{R}$ is a real number.

This is however *not* true anymore, if γ is allowed to depend on the space-time coordinates $\gamma\equiv\gamma(x)$, since

$$\begin{aligned}\partial_\mu\varphi^j(x)\longrightarrow\quad&\partial_\mu\left(e^{i\gamma(x)}\varphi^j(x)\right)\\&=e^{i\gamma(x)}\left(\partial_\mu+i\partial_\mu\gamma(x)\right)\varphi^j(x).\end{aligned} \tag{2.5}$$

The invariance can be restored by introducing a vector field (or: one-form field) $v_\mu(x)$ as an additive contribution to the partial derivative. This gives the following

Definition / Lemma 2. *The* covariant derivative *is defined as*

$$D_\mu \varphi^j(x) = (\partial_\mu + i v_\mu(x))\, \varphi^j(x). \tag{2.6}$$

The Lagrangian

$$\mathcal{L} = -\sum_{i=1}^{n} |D_\mu \varphi^i|^2 - U(\varphi) \tag{2.7}$$

is invariant under the combined gauge transformation

$$\begin{cases} \varphi^i(x) \longrightarrow e^{i\gamma(x)} \varphi^i(x) \\ v_\mu(x) \longrightarrow v_\mu(x) - \partial_\mu \gamma(x). \end{cases} \tag{2.8}$$

Notice: $\mathcal{L} \equiv \mathcal{L}_{\text{kin}}$ defined in (2.7) contains:

- a kinetic term for the φ fields,
- interaction terms between the v_μ and the φ fields,

but *no* kinetic term for the v_μ. Therefore, one could consider v_μ as an auxiliary field and eliminate it using its equations of motion. If v_μ is to be considered as a *physical field*, such as the photon in ordinary QED, we need to add a kinetic term for it into the Lagrangian.

Indeed, the definition

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} \\ \mathcal{L}_{\text{gauge}} &= -\frac{1}{2e^2} v_{\mu\nu} v^{\mu\nu} \end{aligned} \tag{2.9}$$

with $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$ (the *curvature* or *field strength* of the gauge field) gives a gauge-invariant theory where v_μ has a kinetic term.

2.2 Gauge-invariance in supersymmetric QFT

Now we want to mimic this procedure for a *superfield* Φ instead of a scalar field φ. Recall (i.e. from Talk 3 or chapter 12 of [2]) that a $\mathcal{N} = (2, 2)$ chiral superfield in 2 dimensions (which will be our main interest here) has coordinates $x^0, x^1, \theta^\pm, \overline{\theta}^\pm$.

The coordinates $\theta^\pm$ and $\overline{\theta}^\pm$ are *anticommuting*, and hence fulfill $(\theta^\pm)^2 = 0 = (\overline{\theta}^\pm)^2$, so employing a Taylor-expansion-like argument, one can see that Φ can be expanded as

$$\begin{aligned} \Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \varphi &- i\theta^+\overline{\theta}^+\partial_+\varphi - i\theta^-\overline{\theta}^-\partial_-\varphi - \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_+\partial_-\varphi \\ &+ \theta^+\psi_+ - i\theta^+\theta^-\overline{\theta}^-\partial_-\psi_+ + \theta^-\psi_- - i\theta^-\theta^+\overline{\theta}^+\partial_+\psi_- + \theta^+\theta^- F. \end{aligned} \tag{2.10}$$

Here $x^\pm = x^0 \pm x^1$ and $\partial_\pm = \frac{\partial}{\partial x^\pm} = \frac{1}{2}(\partial_0 \pm \partial_1)$ are the derivatives with respect to these coordinates. The fields φ, F and $\psi_\pm$ are fields in ordinary space, i.e. functions of x^μ only.

The equivalent of the theory (2.1) (without potential), is the *manifestly SUSY invariant* Lagrangian $\mathcal{L}$:

$$\mathcal{L} = \int d^4\theta \overline{\Phi}\Phi = \int d\theta^+ d\theta^- d\overline{\theta}^- d\overline{\theta}^+ \overline{\Phi}\Phi. \tag{2.11}$$

The integration with respect to $d^4\theta$ extracts the component of $\overline{\Phi}\Phi$ that contains terms proportional to $\theta^+\theta^-\overline{\theta}^-\overline{\theta}^+$ with appropriate sign.

The Lagrangian (2.11) was chosen such that it admits − just as its scalar field theoretic counterpart (2.1) − a global phase rotation-invariance, that is, $\mathcal{L}$ is unchanged under

$$\Phi \longrightarrow e^{i\alpha}\Phi. \tag{2.12}$$

Now replace α by a *chiral superfield* $A \equiv A(x^\mu, \theta^\pm, \overline{\theta}^\pm)$. Again, since one has $\Phi \longrightarrow e^{iA}\Phi$, the Lagrangian is *not* invariant under this local transformation anymore, since

$$\overline{\Phi}\Phi \longrightarrow \overline{\Phi} e^{-i\overline{A}+iA}\Phi. \tag{2.13}$$

The way out is again the introduction of an auxiliary field V, which is another chiral superfield with appropriate transformation behaviour. This leads to

Lemma 3. *For a chiral superfield V with tranformation behaviour*

$$V \longrightarrow V + i(\overline{A} - A), \tag{2.14}$$

the Lagrangian

$$\mathcal{L}_{kin} = \int d^4\theta \overline{\Phi} e^V \Phi \tag{2.15}$$

is invariant under the combined gauge transformation

$$\begin{cases} \Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) \longrightarrow e^{iA(x^\mu,\theta^\pm,\overline{\theta}^\pm)}\Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) \\ V(x^\mu, \theta^\pm, \overline{\theta}^\pm) \longrightarrow V(x^\mu, \theta^\pm, \overline{\theta}^\pm) + i(\overline{A}(x^\mu, \theta^\pm, \overline{\theta}^\pm) - A(x^\mu, \theta^\pm, \overline{\theta}^\pm)). \end{cases} \tag{2.16}$$

A real superfield with transformation behaviour (2.14) is called a *vector superfield.* One can use the gauge invariance of V to bring its expansion in terms of the $\theta^\pm$, $\overline{\theta}^\pm$ into the form

$$\begin{aligned} V = {} & \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ & + i\theta^-\theta^+(\overline{\theta}^-\overline{\lambda}_- + \overline{\theta}^+\overline{\lambda}_+) + i\overline{\theta}^+\overline{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D. \end{aligned} \tag{2.17}$$

In this expansion, the fields have the following statistics:

- $\lambda_\pm$, $\overline{\lambda}_\pm$ define a *Dirac fermion field*
- D defines a *real scalar field*
- σ, $\overline{\sigma}$ define a *complex scalar field* and
- v_0, v_1 define a *one-form field.*

The gauge where V can be written in the above form is referred to as *Wess-Zumino gauge.*

Notice that there is still a residual gauge symmetry, i.e. gauge transformations that keep the form (2.17), and it is given by

$$v_\mu(x) \longrightarrow v_\mu(x) - \partial_\mu \alpha(x), \tag{2.18}$$

with all other component fields unchanged. Two natural questions arise:

1. *How can $v_{\mu\nu}$ be generalized to a supersymmetric field strength?*
2. *Which of the various fields in the vector multiplet V get kinetic terms, and which are to be eliminated using the equations of motion?*

Definition 4. *For a vector multiplet V, the* super-field strength *is defined as*

$$\Sigma = \overline{D}_+ D_- V. \tag{2.19}$$

Just like $v_{\mu\nu}$, Σ is invariant under the gauge transformation (2.14). The kinetic term for V is given in terms of Σ as

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2e^2}\int d^4\theta \overline{\Sigma}\Sigma. \tag{2.20}$$

By a straightforward, but tedious calculation, one can obtain the component expansions of $\mathcal{L}_{\text{kin}}$ and $\mathcal{L}_{\text{gauge}}$:

$$\mathcal{L}_{\text{gauge}} = \frac{1}{2e^2}(-\partial^\mu \overline{\sigma}\partial_\mu \sigma + i\overline{\lambda}_-(\partial_0 + \partial_1)\lambda_- + i\overline{\lambda}_+(\partial_0 - \partial_1)\lambda_+ + v_{01}^2 + D^2. \tag{2.21}$$

$$\begin{aligned}\mathcal{L}_{\text{kin}} = &-D^\mu \overline{\varphi} D_\mu \varphi + i\overline{\psi}_-(D_0 + D_1)\psi_- + i\overline{\psi}_+(D_0 - D_1)\psi_+ + D|\varphi|^2 + |F|^2 - |\sigma|^2|\varphi|^2 \\ &- \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- - i\overline{\varphi}\lambda_-\psi_+ + i\overline{\varphi}\lambda_+\psi_- + i\overline{\psi}_+\overline{\lambda}_-\varphi - i\overline{\psi}_-\overline{\lambda}_-\varphi.\end{aligned} \tag{2.22}$$

One can also write down so called *twisted F-terms* for Σ. The most important choice for us is

$$\widetilde{W}_{FI,\vartheta} = -t\Sigma = -r\Sigma + i\vartheta\Sigma, \tag{2.23}$$

with $t = r - i\vartheta$, r being called *Fayet-Iliopoulos parameter*[1] and ϑ the *theta angle*. The corresponding contribution to the Lagrangian is

$$\begin{aligned}\mathcal{L}_{FI,\vartheta} &= \frac{1}{2}\left(-t\int d^2\widetilde{\theta}\Sigma + c.c.\right) \\ &= -rD + \vartheta v_{01}.\end{aligned} \tag{2.24}$$

Here, the integration is defined as $d^2\widetilde{\theta} = d\overline{\theta}^- d\theta^+$.

The final result is the Lagrangian for the Gauged Linear Sigma Model

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{FI,\vartheta} + \mathcal{L}_W \\ &= \int d^4\theta \left(\overline{\Phi}e^V\Phi - \frac{1}{2e^2}\overline{\Sigma}\Sigma\right) + \frac{1}{2}\left(-t\int d^2\widetilde{\theta}\Sigma + c.c.\right) + \mathcal{L}_W.\end{aligned} \tag{2.25}$$

[1] hence the subscripts 'FI'.

Comments:

- $\mathcal{L}_W$ is a Lagrangian contribution from a *superpotential* (yet to be introduced).
- D, F have no kinetic term and can be eliminated using the equations of motion.
- After said elimination, one can extract the following potential energy term for σ, φ (neglecting the superpotential):

$$U = |\sigma|^2|\varphi|^2 + \frac{e^2}{2}\left(|\varphi|^2 - r\right)^2. \tag{2.26}$$

3 The different phases of the model

3.1 $\mathbb{CP}^{N-1}$ sigma model (no superpotential)

Consider a $U(1)$ gauge theory with N chiral superfields $\Phi_1, ..., \Phi_N$:

$$\mathcal{L} = \int d^4\theta \left(\sum_{i=1}^{N} \overline{\Phi}_i e^V \Phi_i - \frac{1}{2e^2}\overline{\Sigma}\Sigma\right) + \frac{1}{2}\left(-t\int d^2\widetilde{\theta}\Sigma + c.c.\right). \tag{3.1}$$

After eliminating D and F_i, one obtains again a potential energy term for σ and φ_i:

$$U = \sum_{i=1}^{N} |\sigma|^2|\varphi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{N}|\varphi_i|^2 - r\right)^2. \tag{3.2}$$

From this, one can discuss where U attains 0 for different values of the (real) Fayet-Iliopoulos parameter r:

- $\underline{r > 0}$: $U = 0$ can only be attained if $\sum_{i=1}^{N}|\varphi_i|^2 = r > 0$. Then, $\exists 1 \leq i \leq N$: $|\varphi_i|^2 > 0$, so $\sigma = 0$.
- $\underline{r = 0}$: $U = 0$ attained if $\varphi = 0$, σ arbitrary.
- $\underline{r < 0}$: $U > 0$ for all configurations, so there is *no* zero energy ground state.

For $r > 0$: The set of all classical vacua modulo the $U(1)$ gauge group forms the vacuum manifold, since we require configurations that can be transformed into one another by gauge transformations to be physically equivalent.

In our case, this is

$$\left\{(\varphi_1, ..., \varphi_N)\middle| \sum_{i=1}^{N}|\varphi_i|^2 = r\right\} \Big/ U(1) = \mathbb{CP}^{N-1} \tag{3.3}$$

An analysis of the excitations from the vacuum manifold reveals that in this model, the gauge fields v_μ acquire mass due to the *superHiggs mechanism*, which can be thought of as the supersymmetric generalization of the Higgs mechanism (see [2]).

3.2 Hypersurfaces in $\mathbb{CP}^{N-1}$

Consider a polynomial G of degree d in the variables $\varphi_1, ..., \varphi_N$:

$$G(\varphi_1, ..., \varphi_N) = \sum_{i_1, i_2, ..., i_d} a_{i_1, ..., i_d} \varphi_{i_1} \cdot ... \cdot \varphi_{i_d}. \tag{3.4}$$

Definition 5. *A polynomial* (3.4) *is called* generic *or* transverse *if the following implication holds:*

$$G(\varphi) = \frac{\partial G}{\partial \varphi_1}(\varphi) = ... = \frac{\partial G}{\partial \varphi_N}(\varphi) = 0 \quad \Longrightarrow \quad \varphi_1 = ... = \varphi_N = 0. \tag{3.5}$$

The polynomial G defines the *hypersurface* M of $\mathcal{CP}^{N-1}$ as

$$M = \{\varphi \in \mathbb{CP}^{N-1} | G(\varphi_1, ..., \varphi_N) = 0\}. \tag{3.6}$$

M is a smooth complex manifold with (complex) dimension $N-2$, which justifies its interpretation as a hypersurface.

Consinder a $U(1)$ gauge theory with $N+1$ chiral multiplets $\Phi_1, ..., \Phi_N, P$ such that:

$$\begin{cases} \Phi_1, ..., \Phi_N & \rightsquigarrow U(1)\text{-charge} \quad 1, \\ P & \rightsquigarrow U(1)\text{-charge} \quad -d. \end{cases} \tag{3.7}$$

Then the *GLSM Lagrangian* with *superpotential*

$$W = P \cdot G(\Phi_1, ..., \Phi_N) \tag{3.8}$$

is given by

$$\begin{aligned} \mathcal{L} = \int d^4\theta \Big(\sum_{i=1}^{N} \overline{\Phi}_i e^V \Phi_i + \overline{P} e^{-dV} P - \frac{1}{2e^2} \overline{\Sigma}\Sigma \Big) - \frac{1}{2} \left(\int d^2\widetilde{\theta} \Sigma + c.c. \right) \\ + \frac{1}{2} \left(\int d^2\theta P \cdot G(\Phi_1, ..., \Phi_N) + c.c. \right). \end{aligned} \tag{3.9}$$

From this, one can extract the potential term for the scalar fields as

$$\begin{aligned} U = |\sigma|^2 \sum_{i=1}^{N} |\varphi_i|^2 + |\sigma|^2 d^2 |p|^2 + \frac{e^2}{2} \left(\sum_{i=1}^{N} |\varphi_i|^2 - d|p|^2 - r \right)^2 \\ + \frac{1}{4} |G(\varphi_1, ..., \varphi_N)|^2 + \frac{1}{4} \sum_{i=1}^{N} |p|^2 |\partial_i G|^2. \end{aligned} \tag{3.10}$$

Here, p means the scalar field component of P. The sign of r will determine the structure of the vacuum manifold.

The main analysis will now be to set the right-hand side of equation (3.10) to zero and determine the configurations for σ, p and φ_i that fulfill the equation.

3.2.1 $r > 0$: Calabi-Yau regime

If $r > 0$, $U = 0$ requires $\varphi_i \neq 0$ for one i, therefore $\sigma = 0$.
Assume $p \neq 0$, then $G = \partial_1 G = ... = \partial_N G = 0$. By transversality, one has

$$\varphi_1 = ... = \varphi_N = 0, \tag{3.11}$$

which is a contradiction. It follows that $p = 0$. To sum everything up, $U = 0$ is attained if and only if

$$p = \sigma = 0 \quad \wedge \quad \sum_{i=1}^{N} |\varphi_i|^2 = r \quad \wedge \quad G(\varphi_1, ..., \varphi_N) = 0. \tag{3.12}$$

The vacuum manifold is now the set of all fields satisfying the above equations modulo $U(1)$. This is indeed the hypersurface $M \subseteq \mathbb{CP}^{N-1}$. One can now show, that the requirement $d = N$ makes M a so called *Calabi-Yau manifold.*

Definition 6. *A* Calabi-Yau manifold *is a compact Kähler manifold with vanishing first chern class.*

Theorem 7. *A smooth hypersurface $M \subseteq \mathbb{CP}^{N-1}$ of degree d is a Calabi-Yau manifold if and only if $d = N$.*[2]

The last statement shows that for $r > 0$, the GLSM reduces to a non-linear sigma model on a Calabi-Yau manifold if $N = d$.

3.2.2 $r < 0$: Landau-Ginzburg regime

If $r < 0$, $U = 0$ requires $p \neq 0$, therefore again $\sigma = 0$.
Since $G = \partial_1 G = ... = \partial_N G = 0$, transversality implies $\varphi = 0$, therefore

$$|p| = \sqrt{\frac{|r|}{d}}. \tag{3.13}$$

Any choice of the vacuum, i.e. $\langle p \rangle = \sqrt{|r|/d}$ breaks the gauge invariance, and by the super-Higgs mechanism, the vector multiplets and the P-multiplets gain mass $e\sqrt{|r|/d}$.

If one takes $e \to \infty$, the massive modes decouple from the classical theory and we are left with a theory of the Φ_i fields only, which is a *Landau-Ginzburg theory* with superpotential

$$W = \langle p \rangle G(\Phi_1, ..., \Phi_N). \tag{3.14}$$

References

[1] R. Eager, F. Benini, K. Hori, Y. Tachikawa. *Elliptic genera of 2d N=2 gauge theories.* `arXiv:1308.4896 [hep-th]`

[2] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow. *Mirror Symmetry.* Clay Mathematics Institute, 2003.

[2]The reason is that the first chern class of M is given as $c_1(M) = (N - d)[H]|_M$. See [2].