Seminar on Supersymmetry in Geometry and Quantum Physics

Mathematical Applications of Supersymmetric Quantum Mechanics

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1. Introduction

We saw in the previous talks [**1**][**2**] that there is a strong correspondence between supersymmetric quantum systems like the sigma model and geometric and topological invariants of the space on which it is defined. As one of the most important examples Witten's index was introduced and it was proven that it equals the Euler characteristic of the spacetime manifold.

$$\chi(M) = Tr[(-1)^F e^{-\beta H}] \tag{1.1}$$

The aim of this talk is to derive the theorems from differential geometry by Chern, Gauss & Bonnet, the Hirzebruch signature theorem and Lefschetz fixed-point theorem in the context of certain supersymmetric sigma models on a Riemannian manifold M by computing Witten's index (or slight variations thereof) in terms of a path integral expression and by using the localisation principle. This was originally done in [**6**] and in a more direct way in [**7**] on which this write-up is based. First, we have to discuss the relationship between Witten's index and time evolution in path integral quantisation, especially the appearance of the factor $e^{-\beta H}$.

2. Path Integrals and Time Evolution

We want to derive a path integral expression for the operator $Tr[e^{-\beta H}]$, where H is the Hamiltonian of the system and β some parameter to be fixed later. Consider the partition function of some bosonic field X given by a path integral with fixed boundary conditions

$$Z(X_2,t_2;X_1,t_1)=\int_{X(t_1)=X_1}^{X(t_2)=X_2}\mathscr{D}Xe^{iS[X]}. \tag{2.1}$$

In order to guarantee better convergence properties we will work in Euclidian time, i.e. perform a Wick rotation $t\rightarrow -i\tau$, by which we get

$$Z_E(X_2,\tau_2;X_1,\tau_1)=\int_{X(\tau_1)=X_1}^{X(\tau_2)=X_2}\mathscr{D}Xe^{-S_E[X]}, \tag{2.2}$$

where the subscript "E" indicates Euclidian partition function and Euclidian action respectively.
Let $\mathscr{H}$ be a Hilbert space of states $\mid X_i,\tau_i\rangle$. Then the correlation function is given by (ignoring normalisation)

$$\langle X_2,\tau_2\mid X_1,\tau_1\rangle=\int_{X(\tau_1)=X_1}^{X(\tau_2)=X_2}\mathscr{D}Xe^{-S_E[X]}. \tag{2.3}$$

With this in mind we can interpret the partition function as a (unitary) map (and drop the "E")

$$\begin{aligned}&Z_{\tau_2,\tau_1}:\mathscr{H}\longrightarrow\mathscr{H}\\&f(X_1)\longmapsto f(X_2)=(Z_{\tau_2,\tau_1}f)(X_2):=\int Z(X_2,\tau_2;X_1,\tau_1)f(X_1)dX_1,\end{aligned} \tag{2.4}$$

where f is some functional of the bosonic field variable which in our case will be given by the delta distribution. For the action S_E is invariant under time translation we can write

$$Z(X_2,\tau_2;X_1,\tau_1)=Z(X_2,\tau_2-\tau_1;X_1,0), \tag{2.5}$$

which translates for the above defined maps to

$$Z_{\tau_2,\tau_1}=Z_{\tau_2-\tau_1,0}:=Z_{\tau_2-\tau_1}. \tag{2.6}$$

By evaluation of the integrals one can verify that

$$Z_{\tau_3-\tau_2}\circ Z_{\tau_2-\tau_1}=Z_{\tau_3-\tau_1}, \tag{2.7}$$

or more generally:

$$Z_t\circ Z_{t'}=Z_{t+t'}. \tag{2.8}$$

Therefore we see that the operator Z_t exhibits exactly the same properties as the (Heisenberg-picture) time evolution operator, by which we justify the identification

$$Z_t := e^{-\tau H}, \tag{2.9}$$

in the Heisenberg picture, where H is again the Hamilton operator. To finally make contact with our previously stated aim, consider supersymmetric quantum mechanics with compactified Euclidian time on the smooth manifold S^1_β, the circle of dimension 1 and circumference β. The map between bosonic fields $X(\tau) \mapsto X(\tau + \beta)$ "around the circle" is now given by

$$\int_{X(0)=X_1}^{X(\beta)=X_1} Z(X_1, X_1) dX_1 = Tr[e^{-\beta H}]. \tag{2.10}$$

The corresponding path integral expression is now

$$Z_\beta = \int_{X(\tau+\beta)=X(\tau)} \mathscr{D}X e^{-S_E[X]}, \tag{2.11}$$

i.e. the operator $Tr[e^{-\beta H}]$ can be expressed as a path integral partition function with *periodic* boundary conditions. But this holds *exclusively* for bosonic fields. In order to include fermionic fields into this picture we make use of the following fact known from quantum field theory: Correlation functions in path integral quantisation are *automatically* time ordered. So consider the correlation function of fermionic fields on the circle

$$\langle T\bar{\psi}(\tau_1)\psi(\tau_2)\rangle_{S^1_\beta}. \tag{2.12}$$

Let $0 = \tau_2 < \tau_1 < \beta$ and shift $\tau_2 \to \tau_2 + \beta$. By time translation invariance the following correlation functions have to be equal

$$\langle T\bar{\psi}(\tau_1)\psi(0)\rangle_{S^1_\beta} = \langle T\bar{\psi}(\tau_1)\psi(\beta)\rangle_{S^1_\beta}. \tag{2.13}$$

In path integral quantisation they are given by

$$\int \mathscr{D}\bar{\psi}\mathscr{D}\psi e^{-S_E}\bar{\psi}(\tau_1)\psi(0) = \int \mathscr{D}\bar{\psi}\mathscr{D}\psi e^{-S_E}\psi(\beta)\bar{\psi}(\tau_1), \tag{2.14}$$

so for consistency reasons we have to demand that $\psi(\beta) = -\psi(0)$. This argument easliy generalises to higher correlation functions and we have to impose *anti-periodic* boundary conditions for all path integrals over fermionic field variables. Euclidian time will in the subsequent chapters be reffered to as "t".

3. The Chern-Gauss-Bonnet Theorem

Let M be a compact, oriented, Riemannian manifold of even dimension n without boundary.

Definition 3.1. A *local frame* on $TM = \bigsqcup_{p\in M} T_pM$ is a set of vector fields $\{e_1, ..., e_n\}$ on $U \subset M$ which is linearly independent for all $x \in U$. The Levi-Civita connection on M is then locally given by

$$\nabla_{e_i} e_j = \sum_{k=0}^{n} \Gamma_{ij}^k e_k. \tag{3.1}$$

Let $\{\theta_1, ..., \theta_n\}$ be a dual basis on T^*M in the sense that $\theta_i(e_j) = \delta_{ij}$. Then the *connection form* is defined as

$$\omega_i^j = \sum_{k=0}^{n} \Gamma_{ki}^j \theta^k. \tag{3.2}$$

The *curvature form* on TM is given by

$$\begin{aligned} \Omega_i^j &= d\omega_i^j + \sum_k \omega_k^j \wedge \omega_i^k \\ &= \frac{1}{2}\theta^p \wedge \theta^q R_{pqi}^j, \end{aligned} \tag{3.3}$$

by using the definition of R_{pqi}^j, the Riemann curvature tensor of the Levi-Civita connection.

The result we want to arrive at is the

Theorem 3.1 (Chern-Gauss-Bonnet Theorem). *With the definitions from above we get the equality*

$$\int_M Pf(\Omega) = (2\pi)^n \chi(M), \tag{3.4}$$

Where $Pf(\Omega) = \sqrt{det(\Omega)}$ *is the Pfaffian of the curvature form and* $\chi(M)$ *is the Euler characteristic of* M.

First of all note that the integral is well defined: Ω is a skew-symmetric $n \times n$ matrix with 2-forms as entries so $Pf(\Omega)$ indeed turns out to be a n-form.

We now want to derive this expression in the context of the sigma model: Let (M, g) and S_β^1 be as before. We define a sigma model by

$$\begin{aligned} X : S_\beta^1 &\longrightarrow M \text{ (Bosonic fields)} \\ \text{and } \psi, \bar{\psi} &\in \Gamma(I, X^*TM \otimes \mathbb{C}) \text{ (Fermionic fields),} \end{aligned} \tag{3.5}$$

Its Lagrangian is defined to be

$$\mathscr{L} = \frac{1}{2} g_{ij}(X)\dot{X}^i\dot{X}^j + i g_{ij}(X)\bar{\psi}^i \nabla_t \psi^j - \frac{1}{4} R_{ijkl}(X)\psi^i\psi^j\bar{\psi}^k\bar{\psi}^l \tag{3.6}$$

with

$$\nabla_t \psi^i = \frac{\partial}{\partial t}\psi^i + \Gamma^i_{jk}\dot{X}^j\psi^k. \tag{3.7}$$

Note that this only differs from the expression introduced in [**2**] by a total derivative. Recall, that it enjoys a supersymmetry given by the infinitesimal transformations

$$\begin{aligned} \delta_\epsilon X^i &= \epsilon\bar{\psi}^i - \bar{\epsilon}\psi^i \\ \delta_\epsilon \psi^i &= \epsilon(i\dot{X}^i - \Gamma^i_{jk}\bar{\psi}^j\psi^k) \\ \delta_\epsilon \bar{\psi}^i &= \bar{\epsilon}(-i\dot{X}^i - \Gamma^i_{jk}\bar{\psi}^j\psi^k), \end{aligned} \tag{3.8}$$

where ϵ is an infinitesimal spinor-valued parameter. By starting from the path integral expression for $Tr[e^{-\beta H}]$ we can directly write down the associated one for Witten's index, namely

$$Tr[(-1)^F e^{-\beta H}] = \int_{\text{P.B.C. } \forall \text{ fields}} \mathscr{D}X\mathscr{D}\bar{\psi}\mathscr{D}\psi e^{-S_E}, \tag{3.9}$$

with the Euclidian sigma model action

$$S_E = \int_0^\beta dt \left(\frac{1}{2}g_{ij}(X)\dot{X}^i\dot{X}^j + g_{ij}(X)\bar{\psi}^i\nabla_t\psi^j + \frac{1}{4}R_{ijkl}(X)\psi^i\psi^j\bar{\psi}^k\bar{\psi}^l \right). \tag{3.10}$$

The factor $(-1)^F$ cancels the minus sign in the anti-periodic boundary conditions for fermionic fields, hence we end up with periodic boundary conditions for all fields.
The plan is now to evaluate this path integral. To accomplish this, we have to make use of a powerful tool provided by the presence of supersymmetry: *The localisation principle.* It was introduced in the very first talk [**3**] and its underlying reasons and relationship to equivariant cohomology disscussed in another [**4**]. We will give two short arguments: First a quite interesting one by Witten [**5**] and the one that is normally given and which will actually be more useful.

First Localisation Argument
Consider some arbitrary quantum field theory with a space of functions (fields) $\mathcal{E}$. Let G be its (Super-)Lie Group of continuous symmetries. We will use the following fact:

G acts freely on $\mathcal{E} \Longrightarrow$ all orbits of G are homeomorphic to G.

If this is the case we get a fiber bundle $\mathcal{E} \xrightarrow{\pi} \mathcal{E}\big/G$. The path integral now factorises in the following way.

$$\int_{\mathcal{E}} \mathscr{D}X e^{-S} = \int_G d\mu_G \int_{\mathcal{E}/G} d\mu_{\mathcal{E}/G} e^{-S}, \tag{3.11}$$

with $d\mu_{\mathcal{E}/G}$ beeing some measure on the quotient space and $d\mu_G$ the volume form on the (Super-)Lie Group. If G is a Super-Lie Group, as in the case at hand, the volume form will include Grassmann valued variables and the integral over an argument independent of these Grassmann variables will just yield 0 by the rules of Berezin integration

$$\int_G d\mu_G = 0. \tag{3.12}$$

Hence the whole integral vanishes. In general, however, there is a set of fixed points of the group action $g(\mathcal{E}_0^F) = \mathcal{E}_0^F$ for all $g \in G$ (the "F" denotes "fermionic") where the group does not act freely. Therefore we see that the path integral only receives contributions by an arbitrarily small, G-invariant neighborhood of $\mathcal{E}_0^F$.
In our case, G indeed is a Super-Lie group and this logic holds. Here the fixed points are given by the field configurations for which

$$\begin{aligned} \delta_\epsilon \psi^i &= 0 \\ \delta_\epsilon \bar{\psi}^i &= 0. \end{aligned} \tag{3.13}$$

By looking at the infinitesimal transformations (3.9) this implies $\dot{X} = 0$ and $\Gamma^i_{jk} = 0$, i.e. the path integral localises to constant maps X_0 and the covariant derivative in (3.8) becomes a partial derivative.

Second Localisation Argument
Consider a supersymmetric quantum field theory with supercharge Q generating the symmetry $\delta_\varepsilon \Phi = \varepsilon\{Q, \Phi\}$ for some field Φ. The path integral is given by

$$Z = \int \mathscr{D}\Phi e^{-S[\Phi]}. \tag{3.14}$$

Suppose there exists a fermionic functional $\mathcal{V}$ such that $\{Q, \mathcal{V}\}$ is bosonic and Q-invariant. Then a deformation of the action leads to the partition function

$$Z_t = \int \mathscr{D}\Phi e^{-(S[\Phi]+t\{Q,\mathcal{V}\})} \tag{3.15}$$

for some arbitrary paramerer t. Consider

$$\frac{d}{dt} Z_t = -\int \mathscr{D}\Phi e^{-(S[\Phi]+t\{Q,\mathcal{V}\})}\{Q, \mathcal{V}\} = \int \mathscr{D}\Phi \{Q, e^{-t\{Q,\mathcal{V}\}}\mathcal{V}\} e^{-S[\Phi]} = 0 \tag{3.16}$$

by using the fact that also the action is Q-invariant and that the expectation value of all Q-exact operators must vanish (This also holds in the presence of arbitrary inserted Q-invariant operators in the correlation function). It leads to the conclusion that this construction is actually independent of t, as long as it does not interfere with the convergence of the path integrals, i.e. we assume $Re(t) > 0$. Therefore we might as well take the limit $t \to \infty$ where only those field configurations contribute for which $\{Q, \mathcal{V}\} = 0$.
Lastly we have to note that such a $\mathcal{V}$ almost always exists, take for example

$$\mathcal{V} \sim \{Q^\dagger, \psi\}\bar{\psi}. \tag{3.17}$$

In this case the path integral localises to those configurations for which $\{Q, \psi\} = \delta\psi = 0$ (and therefore also $\delta\bar{\psi} = 0$) as we saw before.

We proceed with the evaluation of the path integral expression for Witten's index but start with outlining the setup[1]:
First of all, Witten's index is independent of β [**1**]. Therefore it is allowed to consider the limit $\beta \to 0$ so that the β-dependent expressions will drop out at the end. By a suitable redefinition of the fields one can accomplish that only the Gaussian approximation of the terms in the action survives, such that the path integrals are Gaussian and quite easy to solve, provided one is able to handle the measure which is generally not the case. The result one arrives at in this way is actually *exact*, which is a common feature of localisation and this is the first huge simplification of the problem due to supersymmetry.
Furthermore, the path integral localises to the constant bosonic maps

$$\begin{aligned} X_0 : & S^i_\beta \longrightarrow M \\ & 0 \longmapsto x. \end{aligned} \tag{3.18}$$

All fields admit a decomposition into Fourier modes due to the periodicity in t

$$\begin{aligned} X^i(t) &= X^i_0 + \hat{X}^i(t) = X^i_0 + \sum_{k\neq 0} a^i_k e^{\frac{2\pi k t i}{\beta}} \\ \psi^i(t) &= \psi^i_0 + \hat{\psi}^i(t) = \psi^i_0 + \sum_{k\neq 0} \xi^i_k e^{\frac{2\pi k t i}{\beta}}, \end{aligned} \tag{3.19}$$

where the index 0 denotes independence of t and the time variation is given by

$$\begin{aligned} \dot{\hat{X}}^i(t) &= \sum_{k\neq 0} a_k \frac{2\pi k i}{\beta} e^{\frac{2\pi k t i}{\beta}} \\ \dot{\hat{\psi}}^i(t) &= \sum_{k\neq 0} \xi_k \frac{2\pi k i}{\beta} e^{\frac{2\pi k t i}{\beta}}. \end{aligned} \tag{3.20}$$

[1]One should probably do this much more rigorously.

Now in the limit $\beta \to 0$ these variations diverge so the corresponding modes get heavily supressed in the path integral due to localisation, except for those configurations where the non constant modes a_k and ξ_k are in an infinitesimal neighbourhood around x, the image of the zero mode. In this case the non zero modes can be arbitrarily good approximated by variables in T_xM. This means that the functional integral over the bosonic fields can be written as an integral over M and an integral over T_xM which is only a flat vectorspace. Additionally all fermion modes are also approximated by tangent space variables by the very definition of the fields (3.5). This is a tremendous simplification as it allows to write the path integral measure as

$$\mathscr{D}X\mathscr{D}\bar{\psi}\mathscr{D}\psi = \sqrt{det(g)}d^nX_0d^n\bar{\psi}_0d^n\psi_0\prod_{k\neq 0}d^na_kd^n\xi_kd^n\bar{\xi}_k. \tag{3.21}$$

This all corresponds to choosing Riemann normal coordinates around x, carrying out the integration of all non-constant modes and fermions and lastly the integration over M.
By plugging the above expressions into the action and recalling that the connection is just given by a partial derivative one arrives at

$$\begin{aligned} S_E = \int_0^\beta dt\Bigg(&\frac{1}{2}g_{ij}\sum_{k\neq 0}\sum_{l\neq 0}a_k^ia_l^j\frac{4\pi^2kl}{\beta^2}e^{\frac{2\pi(k+l)ti}{\beta}} + g_{ij}\sum_{k\neq 0}\sum_{l\neq 0}\xi_k^i\xi_l^j\frac{2\pi li}{\beta}e^{\frac{2\pi(k+l)ti}{\beta}} \\ &+ 2g_{ij}\bar{\psi}_0^i\sum_{k\neq 0}\xi_k^j\frac{2\pi ki}{\beta}e^{\frac{2\pi kti}{\beta}} + \text{ terms proportional to } R_{ijkl}\Bigg). \end{aligned} \tag{3.22}$$

We use the identity

$$\frac{1}{\beta}\int_0^\beta dt\ e^{\frac{\pm i2\pi(k+l)t}{\beta}} = \delta_{k,-l}, \tag{3.23}$$

where δ denotes the Kronecker-Delta and the result is (note that the third term vanishes)

$$\begin{aligned} S_E = \frac{1}{2}g_{ij}\sum_{k\neq 0}a_k^ia_{-k}^j\frac{4\pi^2k^2}{\beta} + g_{ij}\sum_{k\neq 0}\xi_k^i\xi_{-k}^j2\pi ki \\ + \int_0^\beta dt \text{ terms proportional to } R_{ijkl}. \end{aligned} \tag{3.24}$$

Now for the terms containing the Riemann tensor. As we said, the integral is independent of β so we may redefine $\psi_0 \to \beta^{-\frac{1}{4}}\psi_0$, such that β will cancel upon integration of the constant term proportional to $\psi_0\psi_0\bar{\psi}_0\bar{\psi}_0$. The other terms are the one including only t-dependent parts and various combinations. As an instructive example consider the term including $\psi_0\psi_0\hat{\bar{\psi}}(t)\hat{\bar{\psi}}(t)$ which leads to an expression like

$$\psi_0\psi_0\frac{1}{\sqrt{\beta}}\sum_{k\neq 0}\sum_{l\neq 0}\bar{\xi}_k\bar{\xi}_l e^{\frac{2\pi(k+l)ti}{\beta}}. \tag{3.25}$$

Again we can use (3.23) by writing $\frac{1}{\sqrt{\beta}} = \frac{\sqrt{\beta}}{\beta}$ such that the result will be proportional to $\sqrt{\beta}$ and vanish in the limit $\beta \to 0$. Similar arguments can be made for the remaining cases and the only non-vanishing term will be the one containing exclusively the constant modes.
Another redefinition $\hat{X}(t) \to \frac{1}{\sqrt{\beta}}\hat{X}(t)$ leaves us with

$$S_E = \sum_{k\neq 0}\left(\frac{(2\pi k)^2}{2}a_k^i(a_k^i)^* + 2\pi k i\xi_k^i\bar{\xi}_k^i\right) + \frac{1}{4}R_{ijkl}\bar{\psi}_0^i\bar{\psi}_0^j\psi_0^k\psi_0^l + \mathcal{O}(\beta), \tag{3.26}$$

where we identified $a_{-k}^i = (a_k^i)^*$ and $\xi_{-k}^i = \bar{\xi}_k^i$, due to the fields beeing real-valued. The next step is to evaluate this integral with the measure (3.21). Note that the integrals over the non-constant modes are Gaussian (over standard or Grassmann variables)

$$\int e^{-\frac{1}{2}\sum_{k\neq 0}(2\pi k)^2 a_k^i(a_k^i)^*}\prod_{k\neq 0}d^n a_k = \sqrt{\frac{(2\pi)^n}{\prod_{k\neq 0}(2\pi k)^2}} \tag{3.27}$$

for the bosonic variables and

$$\int e^{\sum_{k\neq 0}(2\pi k i)\xi_k^i\bar{\xi}_k^i}\prod_{k\neq 0}d^n\xi_k d^n\bar{\xi}_k^i = \prod_{k\neq 0}(2\pi k) \tag{3.28}$$

for the Grassmann variables. The infinite products cancel exactly and we end up with (we took the prefactor in (3.4) into account)

$$\chi(M) = (2\pi)^{-\frac{n}{2}}\int d(Vol)\int d^n\bar{\psi}_0 d^n\psi_0\; e^{-\frac{1}{4}R_{ijkl}\bar{\psi}_0^i\bar{\psi}_0^j\psi_0^k\psi_0^l}. \tag{3.29}$$

For odd dimensions n of M this integral vanishes, because no term in the series expansion of the exponential function supplies the right amount of ψ's to saturate the Grassmann measure. For even dimension only those right terms contribute and the final result can be shown to be

$$\chi(M) = \frac{(-1)^{\frac{n}{2}}}{2^n(\frac{n}{2})!\pi^{\frac{n}{2}}}\int d(Vol)\;\varepsilon^{i_1 j_1,\ldots,i_m j_m}\varepsilon^{k_1 l_1,\ldots,k_m l_m}R_{i_1 j_1 k_1 l_1}\ldots R_{i_m j_m k_m l_m}, \tag{3.30}$$

with $n = 2m$. This is exactly the coordinate form of the Chern-Gauss-Bonnet formula (3.4).

4. The Hirzebruch-Signature Theorem

Let $H^*(M,\mathbb{Z})$ be the cohomology of M with integer coefficients and let M be closed, oriented, Riemannian and of dimension $4n$. The cup product together with the isomorphism given by Poincaré duality

$$H^{2n}(M,\mathbb{Z}) \times H^{2n}(M,\mathbb{Z}) \xrightarrow{\cup} H^{4n}(M,\mathbb{Z}) \xrightarrow[\sim]{\cap[M]} H_0(M,\mathbb{Z}) \cong \mathbb{Z} \tag{4.1}$$

defines a non-degenerate, symmetric, bilinear form called *intersection form*[2].

DEFINITION 4.1. The *signature* $\sigma(M)$ of M is the signature of the associated quadratic form. As a matrix, it is diagonalisable:

$$\begin{pmatrix} p_1 & & & & & \\ & \ddots & & & 0 & \\ & & p_k & & & \\ & & & n_1 & & \\ & 0 & & & \ddots & \\ & & & & & n_l \end{pmatrix}$$

$\Longrightarrow \sigma(M) = k - l$, for the p's being positiv and the n's being negative.

We want to proof by similar methods

THEOREM 4.1 (Hirzebruch Signature Theorem). *Let M be as above and $\sigma(M)$ be the signature of M. Let Ω be the curvature form on M and let χ_k be the eigenvalues of $\frac{i}{4\pi}\Omega$. Then*

$$\sigma(M) = \int_M \frac{\chi_k}{tanh(\chi_k)} \tag{4.2}$$

First of all note that the integral is well defined. The integrand is called *L-genus* and admits an expansion in terms of the Pontryagin characteristic classes which can be expressed in terms of differential forms. So only those terms with form degree equal to the dimension of M are picked up in the integral.
Consider again the supersymmetric sigma model on M (3.6). The Lagrangian enjoys an additional discrete symmetry given by

$$\bar{\psi} \leftrightarrow \psi. \tag{4.3}$$

Let us call the corresponding operator γ. The first oberservation is that γ sends the ground state to the state which is annihalated by $\bar{\psi}$. To see this recall

[2]In principle such an intersection form can be defined also in other even dimensions, however the case of $dim(M) = 4$ is special as here all the homological information is encoded in $H^2(M)$ and a lot of interesting theorems hold for this case. See e.g. N. Saveliev, "Lectures on the Topology of 3-Manifolds" (There is a chapter about 4-Manifolds, too).

the identification of the Hilbert space of states with the complex of harmonic forms $\Omega^*(M)$ on M from [**2**] and consider

$$\bar{\psi}\gamma \mid 0\rangle = \gamma\psi \mid 0\rangle = 0. \tag{4.4}$$

This state is (with the above identification) essentially given by the volume form

$$\gamma \mid 0\rangle = \underbrace{\sqrt{det(g)}\bar{\psi}_1 ... \bar{\psi}_n \mid 0\rangle}_{d(Vol)}. \tag{4.5}$$

So for a general state $\varphi = \frac{1}{p!}\varphi_{i_1...i_p}\bar{\psi}_{i_1}...\bar{\psi}_{i_p} \mid 0\rangle$ the action of γ is

$$\begin{aligned}\gamma\varphi &= \frac{1}{p!}\varphi_{i_1...i_p}\psi_{i_1}...\psi_{i_p} d(Vol)\\ &= \frac{1}{p!}\varphi_{i_1...i_p}\psi_{i_1}...\psi_{i_p}\frac{1}{n!}\varepsilon_{j_1...j_n}\bar{\psi}_{j_1}...\bar{\psi}_{j_n} \mid 0\rangle\\ &= \frac{1}{p!}\varphi_{i_1...i_p}\frac{1}{(n-p)!}g^{i_1 j_i}...g^{i_p j_p}\varepsilon_{j_1...j_n}\bar{\psi}_{j_p+1}...\bar{\psi}_{j_n} \mid 0\rangle,\end{aligned} \tag{4.6}$$

therefore we will identify $\gamma \equiv *$, where $*$ is the Hodge-star operator. We define a modified form of Witten's index $Tr[*(-1)^F e^{-\beta H}]$.

PROPOSITION 4.1.

$$Tr[*(-1)^F e^{-\beta H}] = \sigma(M) \tag{4.7}$$

PROOF. Lets set $\beta = 0$ in the sequel for this factor plays no role here. We want to evaluate the trace of $*(-1)^F$. The hermitian inner product on the Hilbert space $\Omega^*(M)$ is

$$(\omega, \eta) = \int_M \bar{\omega} \wedge *\eta. \tag{4.8}$$

Let $e_k, k \in \mathbb{N}$ be a Hilbert space basis. Then the trace is given by

$$Tr[*(-1)^F] = \sum_{k\in\mathbb{N}}(e_k, (*(-1)^F)e_k) = \sum_{k\in\mathbb{N}}\int_M \bar{e}_k \wedge *(*(-1)^F)e_k \tag{4.9}$$

Note that the integral is only non zero if $\bar{e}_k \wedge *(*(-1)^F)e_k$ has form degree equal to $dim(M)$, hence in our case only those e_k's that are $2n$-forms contribute. Recall that the Fock space of every supersymmetric quantum theory admits a splitting

$$\mathscr{H} = \mathscr{H}^B \oplus \mathscr{H}^F \tag{4.10}$$

with

$$(-1)^F \mid \varphi\rangle = \begin{cases} 0, & \text{if} \mid \varphi\rangle \in \mathscr{H}^B \\ 1, & \text{if} \mid \varphi\rangle \in \mathscr{H}^F. \end{cases} \tag{4.11}$$

Furthermore double application of the hodge star operator gives back the same form up to a sign

$$**e_k = (-1)^{2n(4n-2n)} e_k = (-1)^{4n^2} e_k = e_k, \tag{4.12}$$

so (4.9) seperates with respect to the eigenvalues of $(-1)^F$

$$\sum_{k\in\mathbb{N},(-1)^F e_k=+1} \int_M \bar{e}_k \wedge e_k - \sum_{k\in\mathbb{N},(-1)^F e_k=-1} \int_M \bar{e}_k \wedge e_k. \tag{4.13}$$

With the well known identification of the intersection form $(\omega \cup \eta) \cap [M]$ with $\int_M \omega \wedge \eta$ in the smooth world, the above expression is the signature of M.

□

With the definitions

$$\psi_+^i = \frac{1}{2}(\bar{\psi}^i + \psi^i) \tag{4.14}$$

$$\psi_-^i = \frac{1}{2}(\bar{\psi}^i - \psi^i) \tag{4.15}$$

the sigma model Lagrangian becomes

$$\mathscr{L} = \frac{1}{2} g_{ij}\dot{X}^i\dot{X}^j + ig_{ij}\psi_+^i\nabla_t\psi_+^j - ig_{ij}\psi_-^i\nabla_t\psi_-^j - \frac{1}{4}R_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l \tag{4.16}$$

and the discrete symmetry via γ is implemented by

$$\begin{aligned} \psi_+^i &\leftrightarrow \psi_+^i \\ \psi_-^i &\leftrightarrow -\psi_-^i. \end{aligned} \tag{4.17}$$

The path integral expression of the modified form of Witten's index is given by

$$Tr[*(-1)^F e^{-\beta H}] = \int_{\text{B.C.}} \mathscr{D}X\mathscr{D}\psi_+\mathscr{D}\psi_- e^{-S_E}, \tag{4.18}$$

where S_E is the associated Euclidian action to (4.16) which is given by

(4.19)

$$S_E = \int_0^\beta dt \left(\frac{1}{2} g_{ij}\dot{X}^i\dot{X}^j + g_{ij}\psi_+^i\nabla_t\psi_+^j - g_{ij}\psi_-^i\nabla_t\psi_-^j + \frac{1}{4}R_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l \right)$$

and the boundary conditions are quite conveniently given by

$$X(\beta) = X(0)$$
$$\psi_+(\beta) = \psi_+(0) \tag{4.20}$$
$$\psi_-(\beta) = -\psi_-(0),$$

as can be seen by exploiting the properties of fermionic correlation functions as above.
As before the path integral localises to constant bosonic field configurations. However, as can be seen from the boundary condition (4.20), there is no constant mode for ψ_-. So the mode expansion looks like

$$\begin{aligned} X^i(t) &= X_0^i + \sum_{k\neq 0} a_k^i e^{\frac{2\pi nit}{\beta}} \\ \psi_+(t) &= \psi_{0,+} + \sum_{k\neq 0} \xi_k^i e^{\frac{2\pi nit}{\beta}} \\ \psi_-(t) &= \psi_{0,-}(t) + \sum_{k\neq 0} \eta_k^i e^{\frac{2\pi nit}{\beta}}. \end{aligned} \tag{4.21}$$

We will rescale similar to the Chern-Gauss-Bonnet case $a_k \to \frac{1}{\sqrt{\beta}}\hat{a}_k$, $\psi_{0,+} \to \sqrt{\frac{2\pi\beta}{i}}\hat{\psi}_{0,+}$ and choose for the t-dependent zero mode of ψ_- the expression $\psi_{0,-} = \frac{1}{2}e^{-\frac{i}{2}\pi+\pi ti}\eta_0^i$ [3].
With these definitions the evaluation in principle proceeds along the same lines as before. However there is another complication due to the definition of ψ_- as we cannot ignore the Christoffel symbol term in the covariant derivate. This can be seen by applying the infinitesimal supersymmetry (3.8) to ψ_-: The Christoffel term drops out and does not vanish upon localisation. Plugging in these mode expansions and using the identity (3.23) leaves us with an action

$$S_E = S_1 + S_2 + \mathcal{O}(\beta) \tag{4.22}$$

with

$$S_1 = \frac{1}{2}\sum_{k\neq 0}(2\pi k)^2 a_k^i (a_k^i)^* + \sum_{k\neq 0}(2\pi ni)\left(\bar{\xi}_k^i \xi_k^i - \bar{\eta}_k^i \eta_k^i\right), \tag{4.23}$$

which is pretty much expected and is obtained in the same way as before and by taking the partial derivative part in (3.7) for the kinetic ψ_- term. The remaining term is a combination of the terms that arise from the second summand in (3.7) and terms proportional to R_{ijkl} by using the expressions in Riemann normal coordinates

$$\Omega_{ij} = R_{ijkl}\psi_0^k\psi_0^l \quad \text{and} \quad \Gamma_{ij}^k = \frac{1}{2}R_{ilj}^k \xi^l. \tag{4.24}$$

[3] I'm not quite sure why this choice is actually allowed. It is neccessary, however, to arrive at the right result at the end.

The result one arrives at is

$$S_2 = -\sum_{k\neq 0} \frac{k}{2}\Omega_{ij}(a_k^i)^* a_k^j + \sum_{k\neq 0}(\frac{i}{2\pi}\Omega_{ij} - \pi i\delta_{ij})\bar{\eta}_k^i\eta_k^j + e^{\frac{i}{2}\pi}(-\frac{i}{4\pi}\Omega_{ij} + \frac{\pi i}{2}\delta_{ij})\bar{\eta}_0^i\eta_0^j. \tag{4.25}$$

The path integrals left to evaluate are all Gaussian. The integration of the non-constant modes proceeds in a similar manner as above and yields a set of determinants which multiply together as

$$det(e^{\frac{i}{2}\pi}(\frac{i}{4\pi}\Omega - \frac{\pi i}{2}id)) \prod_{k\neq 0} \frac{det(id + (\frac{i}{4\pi}\Omega - \frac{\pi i}{2}id)^2)}{det(id + (\frac{\frac{i}{4\pi}\Omega}{k\pi})^2)}. \tag{4.26}$$

We now have to assume that $\frac{i}{4\pi}\Omega$ has eigenvalues χ_k [4]. In this case one can use the identities

$$\begin{aligned} sinh(x) &= x\prod_{k\neq 0}(1 + (\frac{x}{k\pi})^2) \\ cosh(x) &= e^{\frac{i}{2}\pi}(x - \frac{i}{2}\pi)\prod_{k\neq 0}(1 + (\frac{(x - \frac{i}{2}\pi)}{k\pi})^2) \end{aligned} \tag{4.27}$$

and by diagonalising $\frac{i}{4\pi}\Omega$ can rewrite the product of determinants as

$$\prod_{k\neq 0} \frac{\chi_k}{tanh(\chi_k)}. \tag{4.28}$$

It remains to carry out the integrals over the X and ψ_+ zero modes:

$$\sigma(M) = \int d(Vol) \int d^n(\psi_{0,+}) \prod_{k\neq 0} \frac{\chi_k}{tanh(\chi_k)}. \tag{4.29}$$

One can think of the $\psi_{0,+}$'s as a basis of 1-forms on M. The integral projects out the form in the expansion of the integrand which is proportional to $\prod_i \psi_{0,+}^i$, i.e. the top form. So this is exactly the expression from the Hirzebruch Signature Theorem.

5. The Lefschetz Fixed-Point Theorem

Definition 5.1. Let M be a smooth, compact and oriented manifold. Let $f : M \to M$ be smooth and $f_k^* : H^k(M) \to H^k(M)$ be the induced map on the cohomology. If we take real (or rational) coefficients $H^k(M)$ can be thought of as a vector space over $\mathbb{R}$ (or $\mathbb{Q}$) and f_k^* as an endomorphism of vectorspaces. In the

[4] This possibly restricts the set of manifolds on which all of this works.

compact case they are finite dimensional so the trace is well defined and we define the *Lefschetz number*

$$L(f) = \sum_k (-1)^k Tr[f_k^*]. \tag{5.1}$$

Let $p \in M$ be a fixed-point under f and $(Df)_p : T_pM \to T_pM$ be the induced map on the tangent space. The quantity

$$\sigma_p = sgn[det((Df)_p - id_{T_pM})] \tag{5.2}$$

is called *fixed-point index* of p [5]. If p is a fixed-point σ_p gives +1 for preserved orientation and -1 otherwise, i.e. a sum over the fixed-point indices counts the fixed-points with respect to their orientation.

The following remarkable theorem holds:

THEOREM 5.1 (Lefschetz Fixed-Point Theorem). *Let $Fix(f)$ be the fixed-point set of f. Then*

$$\sum_{p \in Fix(f)} \sigma_p = L(f). \tag{5.3}$$

In particular this implies:
If $L(f) \neq 0$ there exists at least one fixed-point of f.

We will proceed as before, but first check how f^* acts on a general (ground-) state $\mid \varphi\rangle \in H^*(M)$.
Such a state is characterized by action of a combination of the operators X^i, ψ^i and $\bar{\psi}^i$ which are given by x^i (coordinates on a chart on M), $g^{ij}\iota_{\frac{\partial}{\partial x^j}} -$ and $dx^i \wedge -$ respectively. Note however, that the interior product

$$\iota_{\frac{\partial}{\partial x^j}}(dx^1 \wedge ... \wedge dx^k) = (-1)^i (dx^1 \wedge ... \wedge \hat{dx}^i \wedge ... \wedge dx^k) \tag{5.4}$$

is only non-vanishing if the right 1-form is already in the wedge product and if it is, it will drop out and only leave a sign. This means every state $\mid \varphi\rangle \in H^*(M)$ is uniquely determined only in terms of action of the operators X and $\bar{\psi}$ and may be written as

$$\mid \varphi\rangle = \sum_{i_1 \dots i_p} \varphi(x)_{i_1 \dots i_p} dx^{i_1} \wedge ... \wedge dx^{i_p}. \tag{5.5}$$

The action of the induced map is then given by

[5] To avoid confusion, there is also another expression which is usually called fixed-point index namely the multiplicity of zeroes of the function $f(x) - x$.

$$f^* \mid \varphi\rangle = \sum_{i_1 \dots i_p} \varphi(f(x))_{i_1 \dots i_p} df(x)^{i_1} \wedge \dots \wedge df(x)^{i_p} \tag{5.6}$$

and in this sense there is no notion of $f^*(\psi)$ where ψ is thought of as an operator. This action corresponds to the transformation of the fields

$$\begin{aligned} X &\longrightarrow f(X) = (f \circ X) : M \to M \\ \bar{\psi} &\longrightarrow f_{\#}(\bar{\psi}) \in \Gamma(I, X^*TM \otimes \mathbb{C}), \end{aligned} \tag{5.7}$$

where $f_{\#}$ is the induced map on $\Gamma(I, X^*TM \otimes \mathbb{C})$. There is again no notion for the transformation of ψ (thought of as a field) as this would induce a map on the cohomology. We need another modified form of Witten's index, namely $Tr[f^*(-1)^F e^{-\beta H}]$.

Proposition 5.1.

$$L(f) = Tr[f^*(-1)^F e^{-\beta H}] \tag{5.8}$$

Proof. We will again set β to zero. Let $e^k; k \in \mathbb{N}$ be a Hilbert space basis of $\Omega^*(M)$ and let $e^p_l; l, p \in \mathbb{N}$ be a Hilbert space basis of $\Omega^p(M)$. In terms of the hermitian inner product on $\Omega^*(M)$ the trace of the operator on $\Omega^*(M)$ is given by

$$Tr[f^*(-1)^F] = \sum_k \int_M \bar{e}_k \wedge *(f^*(-1)^F)e_k = \sum_p \sum_l \int_M \bar{e}^p_l \wedge *(f^p(-1)^F)e^p_l. \tag{5.9}$$

Since e^p_l is a p-form for all l the fermion number operator gives exactly the form degree and we have

$$\sum_p \sum_l \int_M \bar{e}^p_l \wedge *(f^p(-1)^F)e^p_l = \sum_p (-1)^p \sum_l \int_M \bar{e}^p_l \wedge *(f^p)e^p_l = \sum_p (-1)^p Tr[f^p] \tag{5.10}$$

$\square$

A path integral expression for $L(f)$ can be obtained by the follwing consideration. The operator $Tr[(-1)^F e^{-\beta H}]$ describes a propagation along Euclidian time, compactified on S^1_β. The additional application of f^* in $Tr[f^*(-1)^F e^{-\beta H}]$ has the effect of mapping the field space $\mathcal{E}$ on itself by the induced map of the endomorphism f which is given by the action on the fields (5.7). To guarantee consistency one has to demand the boundary conditions

$$\begin{aligned} X(\beta) &= f(X(0)) \\ \bar{\psi}(\beta) &= f_{\#}(\bar{\psi}(0)) \\ \psi(\beta) &= \psi(0) \end{aligned} \tag{5.11}$$

in the path integral

$$L(f) = \int_{B.C.} \mathscr{D}X\mathscr{D}\bar{\psi}\mathscr{D}\psi e^{-S_E}, \tag{5.12}$$

where we will now take the Euclidian sigma model action that was originally introduced in [**2**]

$$S_E = \int_0^\beta dt \left(\frac{1}{2}g_{ij}\dot{X}^i\dot{X}^j + \frac{1}{2}g_{ij}\bar{\psi}^i\nabla_t\psi^j - \frac{1}{2}g_{ij}\nabla_t\bar{\psi}^i\psi^j + \frac{1}{4}R_{ijkl}\psi^i\psi^j\bar{\psi}^k\bar{\psi}^l \right) \tag{5.13}$$

and only differs to the previous one by a total derivative, as already mentioned. There are now two possible ways to argue why the path integral localises additionally to the fixed-point loci under f: The first one is by just looking at the boundary conditions (5.11) where the constant X-modes are exactly fixed-points of f. The second one is the argument originally used in the paper by Alvarez-Gaumé [**6**]:

We have to assume at this point that f is, at least locally, an isometry and has isolated fixed-points $p_i \in Fix(f)$ with $i = 1, ..., m$. In this case the vector field

$$\begin{aligned} K :& M \longrightarrow TM \\ & x \longmapsto (x, K(x)) \quad \text{with} \quad K(x) = \frac{d}{dt}f(x(t)) \end{aligned} \tag{5.14}$$

for some parametrisation $x(t) = (x_1(t), ..., x_n(t))$, is Killing as its flow generates the isometry f. Following the logic in this paper we consider the modified sigma model Lagrangian [6]

$$\begin{aligned} \mathscr{L}_\lambda = \frac{1}{2}g_{ij}\dot{X}^i\dot{X}^j + \frac{i}{2}g_{ij}\bar{\psi}^i\nabla_t\psi^j + \frac{1}{4}R_{ijkl}\bar{\psi}^i\bar{\psi}^j\psi^k\psi^l& \\ - \frac{\lambda^2}{2}g_{ij}K^i(x)K^j(x) - \frac{\lambda}{2}\nabla_i K_j(x)\bar{\psi}^i\gamma_5\psi^j&. \end{aligned} \tag{5.15}$$

It is argued that the path integral must be independent of λ so in the limit $\lambda \to \infty$ it receives only contributions from those configurations where $K(x) = 0$. By looking at the definition of the Killing vector field one can see that these are given by the fixed-points under f as

$$\frac{d}{dt}f(x(t))\bigg|_{x\in Fix(f)} = \frac{df(x)}{dx}\frac{dx(t)}{dt}\bigg|_{x\in Fix(f)} = \frac{d}{dt}x(t)\bigg|_{x\in Fix(f)} \tag{5.16}$$

and we are working in Riemann normal coordinates where $x(t)$ is the origin of its coordinate chart so the initial condition $t = 0$ of the ODE associated to the vector field renders the point t-independent. Therefore we write

[6] It is also supersymmetric and has some interesting connections to Morse Theory but this will not be important here.

$$\int_{B.C.} \mathscr{D}X\mathscr{D}\bar{\psi}\mathscr{D}\psi e^{-S_{E,\lambda}} = \sum_{x\in Fix(f)} \int_{B.C.;\text{ fixed point locus}} \mathscr{D}X\mathscr{D}\bar{\psi}\mathscr{D}\psi e^{-S_E}. \tag{5.17}$$

On the right hand side we will then take the sigma model action (5.13).

Either way, the further evaluation proceeds as always.

Again, we have a mode expansion but the zero-modes are affected by the boundary conditions (5.11). So we have to choose them in such a way that they are consistent. One possible choice is to write for the fields

$$\begin{aligned} X^i(t) &= \frac{1}{\beta}(tf(X_0^i) + (\beta - t)X_0^i) + \sum_{k\neq 0} a_k e^{\frac{2\pi kit}{\beta}} \\ \bar{\psi}(t) &= \left(e^{\frac{t}{\beta}A}\right)^i_j \bar{\psi}_0^j + \sum_{k\neq 0} \bar{\xi}_k e^{\frac{2\pi kit}{\beta}} \\ \psi(t) &= \psi_0 + \sum_{k\neq 0} \xi_k e^{\frac{2\pi kit}{\beta}} \end{aligned} \tag{5.18}$$

where A has to be chosen in such a way that $e^A = (Df)_{x_0}$, where x_0 denotes the image of X_0. It is then straightforward to check that the boundary conditions are fulfilled. First of all, there is no difference to the case of Chern-Gauss-Bonnet when it comes to the non-zero modes. Here the same argument as above holds and the Gaussian integrals together cancel up to an irrelevant prefactor after evaluating the path integral so we only have to care about the zero modes.
For the bosonic zero mode plugging into the action gives

$$\frac{1}{2} g_{ij} \frac{1}{\beta^2} (f(x_0)^i - x_0^i)(f(x_0)^j - x_0^j). \tag{5.19}$$

As the path integral only gets contributions from an arbitrary small neighbourhood around x_0 we might as well take the linear approximation

$$\frac{1}{2} g_{ij} x_0^i \frac{1}{\beta^2} (\partial_i f(x_0)^k - \delta_i^k)(\partial_j f(x_0)^k - \delta_j^k) x_0^j. \tag{5.20}$$

A very similar calculation can be done for the fermionic fields (fortunately one can check that localisation allows us again to only take a partial derivative for the covariant derivative) and after redefinitions $X_0 \to \sqrt{\beta} X_0$, $\bar{\psi}_0 \to \sqrt{2}\bar{\psi}_0$ and $\psi_0 \to \sqrt{2}\psi_0$ the action reads (after integrating over the zero-modes)

$$\begin{aligned} S_E = \frac{1}{2} g_{ij} x_0^i (\partial_i f(x_0)^k - \delta_i^k)(\partial_j f(x_0)^k - \delta_j^k) x_0^j - \bar{\psi}_0^j ((e^A)^i_j - \delta^i_j)\psi_0^i \\ + \int_0^\beta dt \left[\text{non-zero modes} + \mathcal{O}(\beta)\right]. \end{aligned} \tag{5.21}$$

As already mentioned the Gaussian integrals over the non-zero modes cancel. The path integrals over the zero-modes are also Gaussian and give

$$\int e^{-\frac{1}{2}g_{ij}x_0^i(\partial_i f(x_0)^k-\delta_i^k)(\partial_j f(x_0)^k-\delta_j^k)x_0^j}d^n x_0 = \sqrt{\frac{(2\pi)^n}{det((Df)_{x_0}-id_{T_{x_0}M})^T((Df)_{x_0}-id_{T_{x_0}M})}} \tag{5.22}$$

as well as

$$\int e^{\bar{\psi}_0^j((e^A)^i_j-\delta^i_j)\psi_0^i}d^n\bar{\psi}_0 d^n\psi_0 = \int e^{\bar{\psi}_0^j(((Df)_{x_0})^i_j-\delta^i_j)\psi_0^i}d^n\bar{\psi}_0 d^n\psi_0 = det((Df)_{x_0}-id_{T_{x_0}M}). \tag{5.23}$$

Therefore, our final result is

$$L(f) = \sum_{x_0\in F}\frac{det((Df)_{x_0}-id_{T_{x_0}M})}{\sqrt{det((Df)_{x_0}-id_{T_{x_0}M})^T((Df)_{x_0}-id_{T_{x_0}M})}} = \sum_{x_0\in F} sgn\ det((Df)_{x_0}-id_{T_{x_0}M}) = \sum_{x_0\in F}\sigma_{x_0} \tag{5.24}$$

which is exactly the Lefschetz fixed-point formula. Note however, that we omitted a factor of $((2\pi)^{\frac{n}{2}})^2$ that arose due to the Gaussian integrals. As it alters the result only by an overall factor it does not bother us and can be handled by a redefinition of the fixed-point index.

References

1. Menelaos' talk
2. Michael's talk https://www.mathi.uni-heidelberg.de/~walcher/teaching/wise1516/geo_phys/SigmaAndLGModels.pdf
3. Tobias' talk https://www.mathi.uni-heidelberg.de/~walcher/teaching/wise1516/geo_phys/SUSYMathematicalPhysics.pdf
4. Adam's talk https://www.mathi.uni-heidelberg.de/~walcher/teaching/wise1516/geo_phys/eqloc.pdf
5. E. Witten, "Mirror Manifolds and Topological Field Theory", http://arxiv.org/pdf/hep-th/9112056v1.pdf
6. L. Alvarez-Gaumé, "Supersymmetry and the Atiyah-Singer Index Theorem", https://projecteuclid.org/download/pdf_1/euclid.cmp/1103940278
7. S. Li, "Supersymmetric Quantum Mechanics and Lefschetz Fixed-Point Formula", http://arxiv.org/pdf/hep-th/0511101v1.pdf