Seminar on Supergeometry

Sommersemester 2021

Description

Content:

Supergeometry studies varieties characterized by a set of both commuting, or *bosonic*, and anticommuting, or *fermionic*, local coordinates. As such, from a physical point of view, supergeometry lies at the very foundations of supersymmetric theories - such as supergravity and superstring theory -, providing for them a suitable mathematical environment. From a mathematical point of view, supergeometry unfolds notions that force us to revise our classical geometric intuition, leading to genuinely new geometry. Remarkably, it is just where supergeometry diverts from ordinary geometry that the most interesting relations with physics happen. This seminar has a two-fold aim:

1. To introduce to the mathematics of supergeometry and its peculiar aspects, both via abstract general results and theorems, but also via concrete examples and computations.

2. To see supergeometry "in action" in the context of physical supersymmetric theories, thus hinting at its wide range of applications and - possibly - introducing to open research problems.

Audience:

The seminar is aimed at advanced bachelor's or master's students in physics and mathematics. A certain degree of understanding of differential geometry is a pre-requisite for a fruitful participation. A nodding acquaintance with homological algebra and algebraic geometry will help, but it is not to be considered as a pre-requisite. On the physical side, some notions from quantum field theory and supersymmetry might help as well.

Evaluation:

This seminar will be listed in the LSF as "Masterpflichtseminar" in Physics and as "Master-Seminar" in Mathematics. To receive credit in the Physics program, you have to prepare a 6–10 page write-up of your talk, and the final grade will be based (in roughly equal amounts) on both the talk and the writeup. The Math program does not require a write-up, though of course you are welcome to provide one.

Organization

The seminar will meet on *Tuesdays* at 12h (c.t.) via Video. If you were unable to attend the organizational meeting (on March 9) but would like to participate in the seminar, please send us an email.

Organizers and Tutors: Luca Battistella, Simone Noja, Johannes Walcher

Talks

$\mathbb{Z}_2$-graded (Linear) Algebra

Vector superspaces, supermodules and superalgebras, the rule of signs, homomorphisms and matrices in superalgebra. Peculiarities of superalgebra with respect to ordinary algebra, *e.g. supertrace and supertranspose*. Tensor, symmetric and exterior algebras.
Toward homological superalgebra: basic of complexes (*e.g. Koszul & algebraic de Rham complex*) and homological operators.

References: [Del], [Man], [Var], [CCF]

Date: 20 April 2021

Speaker: Niels Gehrigh

Tutor: J.W.

The Berezinian

A landmark of superalgebra and supergeometry. Construction of the *Berezinian* in different ways: *e.g* via linear superalgebra, integral of the supertrace or via homology of the (super) Koszul complex.
Fast forward: Berezinian & Jacobian, a nod to integration on superspaces (*if time permits*).

References: [NoRe], [Del], [Man], [CCF], [HaMM]

Date: 27 April 2021

Speaker: Eugen Dizer

Tutor: S.N.

Smooth Supermanifolds and Batchelor's Theorem

Definition of a *smooth* supermanifolds. In particular, (modern) definition via *ringed spaces*: superspaces as $\mathbb{Z}_2$-graded ringed spaces. First properties of supermanifolds and examples of smooth supermanifolds.
Batchelor's Theorem: any smooth supermanifold is (non-canonically) isomorphic to the total space of a bundle of exterior algebra on an ordinary smooth manifold.
Different approaches to supermanifolds (*if time permits*).

References: [Bat], [Tuy], [Man], [Har], [BBH]

Date: 4 May 2021

Speaker: Johannes Schmidt

Tutor: L.B. & S.N.

INTEGRATION THEORY ON SUPERMANIFOLDS

Definition of *Berezin integral*. Two complexes on a supermanifold: *differential* and *integral* forms. Possibly: relations with *left* and *right* $\mathcal{D}$-modules.
Formal theory: *picture number* and integration "from the total space perspective". *Picture changing operators* and applications in physics.
(No) *Poincaré duality* on supermanifolds: generic sub-supermanifolds and integration. The so-called *pseudoforms*.

References: [Man], [WitInt], [Belo], [Pen], [CNR1]

Date: 11 May 2021

Speaker: Jonathan Paulsen

Tutor: S.N.

SUPERSYMMETRIC LOCALIZATION

Equivariant cohomology, Duistermaat-Heckman and Atiyah-Bott localization, comparison with supersymmetric localization, application to supersymmetric quantum mechanics

References: [AtBo], [GuSt], [Pes],

Date: 18 May 2021

Speaker: Erik Fink

Tutor: J.W.

FUNCTOR OF POINT AND SUPERSCHEMES: TOWARD ALGEBRAIC SUPERGEOMETRY

Yoneda's Lemma and representability: the *functor of points* of a supermanifold. Toward algebraic geometry: superschemes.
Categorial *abstract non-sense* and its relations with physics: supersymmetry transformations and supersymmetry constraints in view of functorial constructions.

References: [Del], [Var], [Lledo], [BrHRPo], [EiHa], [CCGqm]

Date: 25 May 2021?

Speaker: Markus Zetto

Tutor: L.B. & S.N.

COMPLEX SUPERMANIFOLDS: NON-PROJECTED / NON-SPLIT SUPERMANIFOLDS

Real smooth VS complex holomorphic supermanifolds. *Obstruction to splitting / projecting* a supermanifold. General obstruction theory for supermanifolds: a look at *non-abelian cohomology* (possibly).
An easier (but non-trivial) setting: *obstructions for $n|2$-dimensional supermanifolds* and their classification (most probably).
(A lot of) Examples and computations.
Complex Supersymmetric (CS) supermanifolds: superstrings in mathematics.

References: [Man], [NCR2], [Del], [WitSRS], [Green]

Date: 1 June 2021?

Speaker: Jonas Cassel

Tutor: L.B. & S.N.

SUPER RIEMANN SURFACES AND THEIR MODULI

Spin curves & super Riemann surfaces: holomorphic and smooth point of view (following Witten). *Deformations* of super Riemann surfaces: *supermoduli space*.
A hint at supermoduli space and *superstring perturbation theory* in the RNS formalism.

References: [WitSRS], [WitPer], [DonWit], [DHoPhoRev], [DHo]

Date: 8 June 2021?

Speaker: Raphael Senghaas

Tutor: L.B. & S.N.

ODD SYMPLECTIC SUPERMANIFOLDS

Odd Symplectic form and the definition of *odd symplectic supermanifolds*.
Classification of odd symplectic supermanifolds: any odd symplectic supermanifold is the total total space of the (parity shifted) cotangent bundle of an ordinary manifold.
Description of Lagrangian submanifolds.
Toward BV theory: *deformed de Rham complex*, *i.e.* naturality of the *BV Laplacian*.

References: [Mnev], [Sev], [Sch], [JRSW], [QiZa]

Date: 15 June 2021?

Speaker: Johanna Bimmermann / Michale Bleher

Tutor: S.N.

BATALIN-VILKOVISKY (BV) FORMALISM AND SUPERGEOMETRY

Introduction to BV formalism for physical theories. Classical and Quantum BV formalism, master equations and all that (in light of odd symplectic supermanifolds).

References: [Mnev], [Sev], [Sch], [JRSW], [QiZa]

Date: 22 June 2021?

Speaker: Tobias Witt / Ersoy Talha

Tutor: J.W.

CLIFFORD ALGEBRAS AND SPINORS

Definition of *Clifford Algebras* and their *classification*: Bott periodicity.
Spinor Representation: Pin and Spin group.
Spinors and supersymmetries: dimensional reduction.
Spin Manifolds and *Spin Structures*: Stiefel-Whitney classes.
Examples: *Ramond* and *Neveu-Schwarz* spin structures (on S^1) and on genus g compact Riemann surfaces (reprise from spin curves).

References: [BLMi], [AtBoSh], [Var], [Del2]

Date: 29 June 2021?

Speaker: Sebastian Flad

Tutor: L.B. & S.N.

SUPERSPACETIMES: POINCARÉ SUPERGROUP AND MINKOWSKI SUPERSPACE

A nod to *Lie supergroups* and *Lie superalgebras*: classification of Lie superalgebras. *Coleman-Mandula* and *Haag-Łopuszańksi-Sohnius* theorems.
The notion of *superspacetime*: the case of Minkowski superspace.
Supersymmetric Lagrangians on "flat" superspacetimes: examples for different amount of supersymmetries.
Toward *Supergravity* and "curved" superspacetimes (*if time permits*).

References: [Kac], [Var], [CCF], [FSS], [Freed], [CoMa], [HaLoSo], [CdAF] [DelFre], [CCG1], [CCG2], [CCG3]

Date: 6 July 2021?

Speaker: Simon Brunner / Andrea Grossutti

Tutor: J.W. & S.N.

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