# Functor of Points and Superschemes

**Towards algebraic Supergeometry**

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## Contents



# 1 The Yoneda-Lemma

We start with a short reminder of several essential definitions in category theory - feel free to skip it or skim for the notation.

**Definition 1.1.** A *category* $\mathcal{C}$ is an algebraic construction consisting of the following pieces of data:

- A collection of objects $ob\,\mathcal{C}$, that we will also just denote by $\mathcal{C}$
- For $C, D \in \mathcal{C}$ a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(C, D)$
- For $C, D, E \in \mathcal{C}$ a composition operation

$$\circ : \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{C}}(C, E)$$

- For $C \in \mathcal{C}$, an identity morphism $\mathrm{id}_C$.

We demand composition to me associative, and composition with the identity on both sides to not change a morphism.

**Example 1.2.**
- *The category* Set*, with objects being (small) sets and morphisms being arbitrary functions*
- *The category* CRing *of commutative rings (with unit!) with ring homomorphisms*
- *The category* SRing *of superrings (i.e. graded commutative $\mathbb{Z}_2$-graded rings) with graded ring homomorphisms*
- *The category* Mfd *of manifolds and smooth maps*
- *The category* SMfd *of supermanifolds and maps of locally superringed spaces*
- *For $\mathcal{C}$ an arbitrary category, the opposite category $\mathcal{C}^{op}$ with $\mathrm{Hom}_{\mathcal{C}^{op}}(C, D) = \mathrm{Hom}_{\mathcal{C}}(D, C)$ and composition turned around - note this is purely a formal rewriting, and we do not need to impose invertibility on the morphisms or anything like that.*

**Definition 1.3.** A morphism $f : C \to D$ in a category $\mathcal{C}$ is called *isomorphism* if it possesses an inverse $g : D \to C$, i.e. satisfying $f \circ g = \mathrm{id}_D$ and $g \circ f = \mathrm{id}_C$.

**Definition 1.4.** Let $\mathcal{C}, \mathcal{D}$ be categories, then a *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A map $F : ob\, \mathcal{C} \to ob\, \mathcal{D}$ between objects
- For $C, C' \in \mathcal{C}$ a map $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(D))$

such that $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ and $F(g \circ f) = F(g) \circ F(f)$ hold.

**Definition 1.5.** For $F, G : \mathcal{C} \to \mathcal{D}$ two functors, a *natural transformation* $\eta : F \Rightarrow G$ is given by a collection of morphisms $\eta_C : F(C) \to G(C)$ for every $C \in \mathcal{C}$, making the following square commute for every $f : C \to C'$ in $\mathcal{C}$.

$$\begin{tikzcd} F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\eta_{C'}"'] & G(C') \end{tikzcd}$$

**Definition 1.6.** For categories $\mathcal{C}$ and $\mathcal{D}$, we define the *functor category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ consisting of objects that are functors from $\mathcal{C}$ and $\mathcal{D}$, and morphisms that are natural transformations between such functors - check that this is indeed a category.

**Definition 1.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We call it:

- *essentially surjective* if its essential image, i.e. the set of objects in $\mathcal{D}$ that are isomorphic to an object $F(C)$ with $C \in \mathcal{C}$, is all of $\mathcal{D}$
- *fully faithful* if the induced map $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ is a bijection for all $C, C' \in \mathcal{C}$
- *equivalence* if it is both of the above.

**Theorem 1.8.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if there is another functor $G : \mathcal{D} \to \mathcal{C}$, such that $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.*

*Remark.* Compare this to the statement that a map of sets is a bijection iff it is invertible.

**Lemma 1.9.** *Let $F : \mathcal{C} \hookrightarrow \mathcal{D}$ be fully faithful (we notate this by a hooked arrow) and $\mathcal{E}$ be its essential image, then $F$ induces an equivalence $\mathcal{C} \simeq \mathcal{E}$, so we can identify $\mathcal{C}$ with a full subcategory of $\mathcal{D}$ (by this we mean a subset of objects of $\mathcal{D}$ together with all morphisms between them - this is again a category).*

*Proof.* Since the corestriction $F|^{\mathcal{E}} : \mathcal{C} \to \mathcal{E}$ exists, is still fully faithful and by definition of $\mathcal{E}$ essentially surjective, this follows by the definitions. $\square$

With all of this machinery at hand, we state the famous *Yoneda Lemma*:

**Theorem 1.10** (Yoneda-Lemma). *For any category $\mathcal{C}$, the functor*

$$\begin{aligned} j_{\mathcal{C}} : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{op}, Set) \\ C &\mapsto \mathrm{Hom}_{\mathcal{C}}(-, C) \\ (f : C \to D) &\mapsto (f \circ - : \mathrm{Hom}_{\mathcal{C}}(-, C) \to \mathrm{Hom}_{\mathcal{C}}(-, D)) \end{aligned} \tag{1}$$

*is fully faithful. We call it the Yoneda embedding, and we further denote* $\mathrm{Fun}(\mathcal{C}^{op}, Set) =:$ *$\mathcal{PSh}(\mathcal{C})$ the presheaf category of $\mathcal{C}$.*

*Proof.* It is a good exercise to check that the above construction is indeed well-defined and a functor; we will not further elaborate this. To prove that it is fully faithful, we need to show that for any $C, C' \in \mathcal{C}$, the map $\Psi : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{PSh}(\mathcal{C})}(j(C), j(C'))$ induced by $j_{\mathcal{C}}$ is a bijection. For this, construct an inverse

$$\begin{aligned} \Phi : \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, C), \mathrm{Hom}_{\mathcal{C}}(-, C')) &\to \mathrm{Hom}_{\mathcal{C}}(C, C') \\ \eta &\mapsto \eta_C(\mathrm{id}_C) \end{aligned} \tag{2}$$

We now go on to check that these maps are indeed inverse to each other:

$$\Phi \circ \Psi(f) = \Psi(f)_C(\mathrm{id}_C) = f \circ \mathrm{id}_C = f \tag{3}$$

$$(\Psi \circ \Phi(\eta))_X(g) = (\Psi(\eta_C(\mathrm{id}_C)))_X(g) = \eta_C(\mathrm{id}_C) \circ g = \eta_X(g) \tag{4}$$

The last equality follows by chasing $\mathrm{id}_C$ around the following naturality square:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, C) \arrow[r, "\eta_C"] \arrow[d, "- \circ g"] & \mathrm{Hom}_{\mathcal{C}}(C, C') \arrow[d, "- \circ g"] \\ \mathrm{Hom}_{\mathcal{C}}(X, C) \arrow[r, "\eta_X"] & \mathrm{Hom}_{\mathcal{C}}(X, C') \end{tikzcd}$$

$\square$

By 1.9, this means that we may regard $\mathcal{C}$ (up to equivalence) as a full subcategory of $\mathcal{PSh}(\mathcal{C})$. We call presheaves that lie in the essential image of the Yoneda embedding *representable.*

**Corollary 1.11.** *An object $C \in \mathcal{C}$ is, up to isomorphism, uniquely determined by all maps into it. More formally, we claim that for any other $C' \in \mathcal{C}$, we have that*

$$\mathrm{Hom}_{\mathcal{C}}(-, C) \cong \mathrm{Hom}_{\mathcal{C}}(-, C') \tag{5}$$

*are naturally isomorphic if and only if already $C \cong C'$.*

*Proof.* Note that by 1.9, the essential image of the Yoneda embedding is equivalent to $\mathcal{C}$. Since functors always preserve isomorphisms (as they preserve composition and identities), our statement follows from Theorem 1.8 which gives us an inverse to this equivalence. $\square$

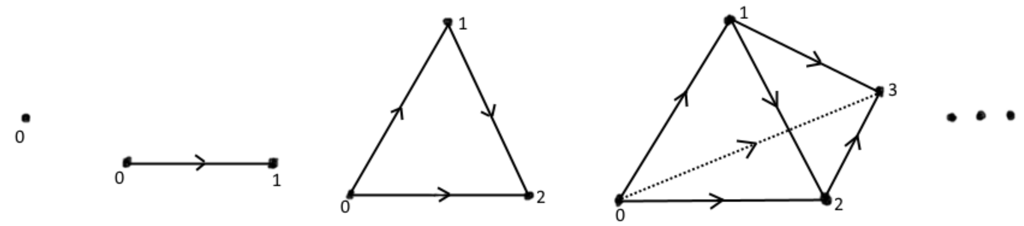
*Remark.* There is an analogous dual version of the Yoneda Lemma, abstractly expressing the similar statement that an object $C \in \mathcal{C}$ is determined up to isomorphism by the morphisms *out of* it. Further, there is a more general statement going by the same name: For $C \in \mathcal{C}$ and functors $F : \mathcal{C}^{op} \to \mathrm{Set}$ and $F' : \mathcal{C} \to \mathrm{Set}$, there are natural (in any possible way) isomorphisms:

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, C), F) \cong F(C) \tag{6}$$

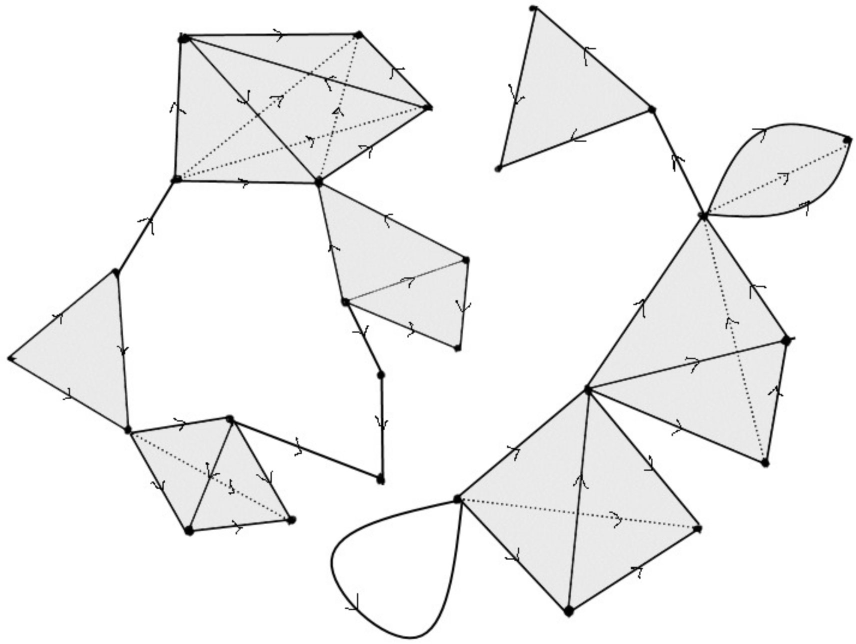
$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(C, -), F') \cong F'(C) \tag{7}$$

The reader might feel that all of these are purely esoteric considerations, but we will argue that the Yoneda-Lemma has a deep and very figurative meaning:

**Example 1.12.** *Let $\Delta$ be the category of simplices, with objects the nonempty finite totally ordered sets $[n] = 0 < 1 < \cdots < n$ for $n \in \mathbb{N}_0$, and morphisms being order-preserving maps (i.e. $x \leq y$, then $f(x) \leq f(y)$). We imagine the objects of this category as $n$-dimensional tetrahedra:*




*But how does the presheaf category $\mathcal{PSh}(\Delta) := \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$ look like? Such a functor $X_\bullet$ maps each natural number $n$ to a set $X_n$, which we call the set of $n$-simplices in $X_\bullet$, and by working out the combinatorics of possible maps in $\Delta$, we realize that the simplices are glued together along their edges, forming so-called simplicial sets - a more general form of simplicial complexes. An example would be:*



*The Yoneda Lemma has a straightforward interpretation in this setting: It just states that $n$-simplices are already simplicial sets by themselves, i.e. $\Delta$ is a full subcategory of the category of simplicial sets $\mathcal{PSh}(\Delta)$.*

We thereby note that presheaves on a category $\mathcal{C}$ can be interpreted as *formal gluings* of objects in $\mathcal{C}$. The geometric significance of this gets even more apparent if we turn to differential geometric constructions:

**Question 1.13.** *Above statements tell us that a manifold is uniquely determined by either of the functors*

$$C^\infty(-, M) \qquad C^\infty(M, -) \tag{8}$$

*namely the smooth maps from other manifolds into or out of it - the same holds for supermanifolds. We call the first functor, represented by $M$, the* ***functor of points*** *of $M$. Morphisms from a (super-)manifold $S$ into $M$ are then often called $S$-points of $M$; they are the possible ways of laying $S$ out in $M$, i.e. probing our manifold with $S$.*

*This idea is commonly used when defining new (super-)manifolds: One explicitly writes down such a contravariant functor and then shows that it is representable. But giving such a functor is a lot of information to carry around - can we do better?*

For this, let Cart be the category of Cartesian spaces $\mathbb{R}^n$, together with smooth maps between them. Intuitively, glueing these together in *good* ways, combined with certain finiteness conditions, should give us smooth manifolds - by the above heuristic, we would therefore wish for the Yoneda embedding to factor as

$$\mathrm{Cart} \hookrightarrow \mathrm{Mfd} \hookrightarrow \mathcal{PSh}(\mathrm{Cart})\ ; \tag{9}$$

thereby establishing manifolds as a full subcategory of presheaves on Cart.

*Proof.* Let $M, N$ be smooth manifolds of dimensions $m, n$. We want to show that the map $\mathrm{Hom}_{\mathrm{Mfd}}(M,N) \to \mathrm{Hom}_{\mathcal{PSh}(\mathrm{Cart})}(C^\infty(-,M), C^\infty(-,N))$ induced by postcomposition has an inverse. Namely, given a natural transformation $\eta$ in the right hand side, we want to obtain a corresponding smooth map $\phi : M \to N$.

For this, choose an atlas $M = \bigcup U_i$, let $\iota_i : U_i \hookrightarrow M$ denote the inclusion and let $\phi_i := \eta_{U_i}(\iota_i) : U_i \to N$. If there indeed exists such a $\phi$, then $\phi|_{U_i}$ must be equal to $\phi_i$ as can be seen by noting that $\eta_{U_i}$ must be given by postcomposition with $\phi$.

These $\phi_i$ fit together to glue a smooth map $\phi : M \to N$, which is thereby also unique. One can see this by noting that the intersection $U_i \cap U_j$, while not being a chart itself, can be covered by charts $W_{ijk}$ diffeomorphic to $\mathbb{R}^n$. We construct $\phi_{ijk} : W_{ijk} \to N$ in the same way as above, and use the naturality square below to show that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ since their restrictions to eack $W_{ijk}$ coincide.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Mfd}}(U_i, M) \arrow[r, "\eta_{U_i}"] \arrow[d] & \mathrm{Hom}_{\mathrm{Mfd}}(U_i, N) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Mfd}}(W_{ijk}, M) \arrow[r, "\eta_{W_{ijk}}"] & \mathrm{Hom}_{\mathrm{Mfd}}(W_{ijk}, N) \\
\mathrm{Hom}_{\mathrm{Mfd}}(U_j, M) \arrow[r, "\eta_{U_j}"] \arrow[u] & \mathrm{Hom}_{\mathrm{Mfd}}(U_j, N) \arrow[u]
\end{tikzcd}$$

Alternatively, compatibility of the $\phi_i$ can also be deduced from (6), which implies that the $\phi_{ij}$ are uniquely determined by the induced transformations in $\mathrm{Hom}_{\mathcal{PSh}(\mathrm{Cart})}(C^\infty(-,W_{ijk}), C^\infty(-,N))$ (try to elaborate this yourself!). $\square$

*Remark.* There is a more conceptual, alternative proof of this statement using the *Comparison Lemma* from Topos Theory - this is essentially just a categorification of our proof of Lemma 2.4.

**Theorem 1.14.** *A smooth manifold $M$ is uniquely determined by either:*

- *The induced functor $C^\infty(-,M) : \mathrm{Cart}^{op} \to \mathrm{Set}$, i.e. we obtain a fully faithful functor $\mathrm{Mfd} \hookrightarrow \mathcal{PSh}(\mathrm{Cart})$*
- *The $\mathbb{R}$-algebra of smooth $\mathbb{R}$-valued functions $C^\infty(M,\mathbb{R})$, i.e. we obtain a fully faithful functor $\mathrm{Mfd} \hookrightarrow \mathrm{CAlg}_{\mathbb{R}}^{op}$*

*Similarly, a smooth supermanifold $(M, \mathcal{O}_M)$ is uniquely determined via*

- *The functor $\mathrm{Hom}_{\mathrm{SMfd}}(-,M) : \mathrm{SCart}^{op} \to \mathrm{Set}$*
- *The graded commutative $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra of global sections $\mathcal{O}_M(M)$*

*where $\mathrm{SCart}$ is the category of super-Cartesian spaces $\mathbb{R}^{p|q}$ together with maps of locally superringed spaces.*

*Proof.* The first statement was already discussed above, and the statement about supermanifolds can be deduced in a completely analogous way. The fact that a smooth manifold is determined by its smooth functions is a deep theorem in differential geometry, known as *Milnor's exercise*, that we can't do justice here - we refer to Chapter 35 in [KMS13]. The analogous claim for supermanifolds follows easily, see [HST11]. □

*Remark.* On super-Cartesian spaces, this last embedding into graded algebras is explicitly given via:

$$\mathbb{R}^{p|q} \mapsto \mathrm{C}^\infty(\mathbb{R}^p) \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^q \tag{10}$$

We shortly summarize what we have found (the inclusions of presheaf categories follow by abstract nonesense on Left Kan Extensions):

$$\begin{tikzcd}
 & \mathrm{CAlg}_{\mathbb{R}}^{op} & & \\
\mathrm{Cart} \arrow[ur, hook] \arrow[r, hook] \arrow[d, hook] & \mathrm{Mfd} \arrow[u, hook] \arrow[r, hook] \arrow[d, hook] & \mathcal{PS}h(\mathrm{Cart}) \arrow[r, hook] \arrow[d, hook] & \mathcal{PS}h(\mathrm{Mfd}) \arrow[d, hook] \\
\mathrm{SCart} \arrow[r, hook] \arrow[dr, hook] & \mathrm{SMfd} \arrow[r, hook] \arrow[d, hook] & \mathcal{PS}h(\mathrm{SCart}) \arrow[r, hook] & \mathcal{PS}h(\mathrm{SMfd}) \\
 & \mathrm{SAlg}_{\mathbb{R}}^{op} & &
\end{tikzcd}$$

**Example 1.15.** *Above statement is extremely helpful when studying spaces of fields:*

- *Let $M$ be an ordinary manifold, then:*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SMfd}}(\mathbb{R}^{0|1}, M) &\cong \mathrm{Hom}_{\mathrm{SAlg}_{\mathbb{R}}}\left(C^\infty(M), \bigwedge^* \mathbb{R}\right) \cong \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{R}}}\left(C^\infty(M), \mathbb{R}\right) = \\ &= \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{R}}}\left(C^\infty(M), C^\infty(\mathbb{R}^0)\right) \cong \mathrm{Hom}_{\mathrm{Cart}}(\mathbb{R}^0, M) \cong M\end{aligned}$$

  *Note that these smooth functions correspond to the evaluation maps $\mathrm{ev}_v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ that send $\alpha \mapsto \alpha(v)$, for $v \in \mathbb{R}^n$.*

- *Together with a standard fact from differential geometry, this helps us find:*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SMfd}}(\mathbb{R}^{0|2}, M) &\cong \mathrm{Hom}_{\mathrm{SAlg}_{\mathbb{R}}}\left(C^\infty(M), \bigwedge^* \mathbb{R}^2\right) \overset{\epsilon=\theta_1\theta_2}{\cong} \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{R}}}\left(C^\infty(M), {}^{\mathbb{R}[\epsilon]}\!\big/\!_{(\epsilon^2)}\right) = \\ &= \{(f,g) : C^\infty(M) \to \mathbb{R} | f \text{ algebra hom, } g(\alpha_1\alpha_2) = f(\alpha_1)g(\alpha_2) + g(\alpha_1)f(\alpha_2)\} = \\ &= \left\{f \in \mathrm{Hom}_{\mathrm{Cart}}(\mathbb{R}^0, M) \cong M, g : C^\infty(M) \to \mathbb{R} | g(\alpha_1\alpha_2) = \alpha_1(f)g(\alpha_2) + g(\alpha_1)\alpha_2(f)\right\} = \\ &\cong TM\end{aligned}$$

- *We claim that* $[\mathbb{R}^{0|1}, M] := \mathrm{Hom}_{\mathrm{SMfd}}(-\times\mathbb{R}^{0|1}, M) \cong \mathrm{Hom}_{\mathrm{SMfd}}(-, \Pi TM) \in \mathcal{PSh}(\mathrm{SMfd})$, *namely is representable by the odd tangent bundle. This seems reasonable, since:*
$$[\mathbb{R}^{0|1}, M](\mathbb{R}^{0|0}) = \mathrm{Hom}_{\mathrm{SMfd}}(\mathbb{R}^{0|1}, M) \overset{above}{\cong} M$$
$$[\mathbb{R}^{0|1}, M](\mathbb{R}^{0|1}) = \mathrm{Hom}_{\mathrm{SMfd}}(\mathbb{R}^{0|2}, M) \overset{above}{\cong} TM$$
- *Finally we define the space of superfields on* $M$ *with values in* $\mathbb{R}^{p|q}$ *as the functor* $[M, \mathbb{R}^{p|q}] := \mathrm{Hom}_{\mathrm{SMfd}}(- \times M, \mathbb{R}^{p|q}) \in \mathcal{PSh}(\mathrm{SMfd})$. *Note that this presheaf is generally not representable - if* $M$ *is compact, we can however understand it as a Fréchet manifold.*

**Exercise 1.16.** *Use the above methods, as well as 3.1, to evaluate:*

1. $\mathrm{Hom}_{\mathrm{SMfd}}(\mathbb{R}^{n|0}, \mathbb{R}^{0|b})$
2. $\mathrm{Hom}_{\mathrm{SMfd}}(M, \mathbb{R}^{n|m})$ *for* $M$ *a manifold*
3. $\mathrm{Hom}_{\mathrm{SMfd}}(\mathbb{R}^{0|m}, M)$ *for* $m = 0, 1, 2, 3$ *and* $M$ *a manifold*
4. $\mathrm{Hom}_{\mathrm{SMfd}}(\mathbb{R}^{0|a}, \mathbb{R}^{0|b})$ *is a bit more difficult*

*Finally, for* $R, S, T$ *arbitrary supermanifolds (or even presheaves on* SMfd*), try to make sense of the construction* $[S, T] \in \mathcal{PSh}(\mathrm{SMfd})$ *and show (using 6):*

$$\mathrm{Hom}_{\mathcal{PSh}(\mathrm{SMfd})}\left(j_{\mathrm{SMfd}}(R), [S,T]\right) \cong \mathrm{Hom}_{\mathrm{SMfd}}(R \times S, T) \cong \mathrm{Hom}_{\mathcal{PSh}(\mathrm{SMfd})}\left(j_{\mathrm{SMfd}}(S), [R,T]\right)$$
$$[R, [S,T]] \cong [R \times S, T] \cong [S, [R,T]]$$

# 2 Schemes and Superschemes

We have seen that (smooth!) manifolds and supermanifolds can be recovered by their (super-)algebra of functions. Thus, given an arbitrary (super-)commutative algebra, we might think about somehow constructing a space such that elements of our algebra are exactly the functions on this space. We will now see that this is indeed possible, albeit yielding a different and conceptually new kind of space - a (super-)scheme.

## 2.1 Schemes

**Definition 2.1.** Let $R$ be a commutative ring, then a *multiplicative subset* $S \subseteq R$ is a collection of elements such that $1 \in S$ and for $a, b \in S$, also $a \cdot b \in S$.

In this case, we can define a new commutative ring $R[S^{-1}]$ that consists of elements $\frac{r}{s}$ with $r \in R$, $s \in S$, where we identify two such elements iff

$$\frac{r}{s} = \frac{r'}{s'} \quad \Leftrightarrow \quad \exists\, t \in S : t \cdot (rs' - r's) = 0\ . \tag{11}$$

We give this set of (equivalence classes of) fractions the usual addition and multiplication from fractional algebra, and call the obtained ring the *localization* of $R$ at $S$.

**Example 2.2.**
- *If $R = \mathbb{Z}$ and $S = \mathbb{Z}^\times$, this gives us the usual construction of $\mathbb{Q}$.*
- *If $0 \in S$, then by setting $t = 0$ above we immediately find $R[S^{-1}] = 0$.*
- *For $s \in S$ nilpotent (e.g. odd element of a superring), $0 = s^N \in S$ so $R[S^{-1}] = 0$.*
- *We have a natural map $R \to R[S^{-1}]$ sending $r \mapsto \frac{r}{1}$, similarly for $S \subset T$ multiplicative subsets we get a map $R[S^{-1}] \to R[T^{-1}]$.*
- *For $f \in R$, the set $\{1, f, f^2, f^3, \dots\}$ is multiplicatively closed, and we call the localization of $R$ at it $R_f$.*
- *For $\mathfrak{p} \trianglelefteq R$ a prime ideal, $R - \mathfrak{p}$ is multiplicatively closed by definition, and we call the localization at it $R_\mathfrak{p}$.*

Before we use this definition to construct affine schemes, let us first introduce a workhorse lemma that will simplify things a lot.

**Definition 2.3.** Let $X$ be a topological space and $\mathcal{B}$ a basis of the topology. We call this basis *stable under intersections* or IS if for $U, V \in \mathcal{B}$ also $U \cap V \in \mathcal{B}$. Further, we let $\mathrm{Open}(X)$ denote the category of open sets in $X$ and $\mathrm{Open}(\mathcal{B})$ the category of of open sets in $\mathcal{B}$; with morphisms given by inclusions of open subsets.

**Lemma 2.4** (0-comparison lemma)**.** *Let $X$ be a topological space and $\mathcal{B}$ be an IS basis of $X$. Then, the following data are equivalent:*
- *A sheaf $\mathcal{F} : \mathrm{Open}(X)^{op} \to \mathrm{Set}$ on $X$*
- *A sheaf $\mathcal{F} : \mathrm{Open}(\mathcal{B})^{op} \to \mathrm{Set}$ on the basis $\mathcal{B}$ - since it is IS, we can just formalize the sheaf axioms only on sets in this basis.*

*Proof Sketch.* We have an obvious inclusion functor $\iota : \mathrm{Open}(\mathcal{B}) \hookrightarrow \mathrm{Open}(X)$, which by precomposition induces a functor on presheaf categories

$$\iota^* : \mathrm{Fun}(\mathrm{Open}(X)^{op}, \mathrm{Set}) \to \mathrm{Fun}(\mathrm{Open}(\mathcal{B})^{op}, \mathrm{Set}) \tag{12}$$

It follows from the definitions that this functor restricts to the actual categories of sheaves, i.e. if $F$ is a sheaf on $X$ then restricting it to open sets in $\mathcal{B}$ gives us a sheaf an $\mathcal{B}$. Our goal is to show that $\iota^*$ is always an equivalence, and by theorem 1.8 this can be done by giving an inverse functor

$$\mathrm{Ran} : \mathcal{S}h(\mathcal{B}) \to \mathcal{S}h(X)\ . \tag{13}$$

For a sheaf $F$ and $U \subseteq X$ open, choose a cover $U = \bigcup U_i$ by $U_i \in \mathcal{B}$ and define:

$$\mathrm{Ran}\, F(U) := \left\{ (x_i) \in \prod F(U_i) \,|\, \forall i, j : x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j} \right\} \tag{14}$$

It is easy (and instructive) to check that this is indeed independent of the chosen covering, actually a sheaf, and that it is the inverse functor to $\iota^*$ we are searching for. $\square$

*Remark.* We have given Ran this strange name for a reason, it is actually the so-called *Right Kan Extension* functor along $\iota$. Noticing this, and some abstract nonsense, makes almost all of this proof trivial. Further, the way we have named this lemma is due to a categorification of it being well-known in topos theory as the *comparison lemma*.

With this knowledge, we are fit to define affine schemata, which should build a geometric incarnation of the usual theory of rings; and quasicoherent modules that can be understood in analogy to modules over rings.

**Definition 2.5** (Zariski-Topology)**.** Let $R$ be a commutative ring, then denote by $\mathrm{Spec}(R)$ the set of prime ideals in that ring. We define a topology on this set by giving its closed subsets: These should be exactly the subsets of the form

$$\mathrm{V}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} \supseteq \mathfrak{a}\} \tag{15}$$

where $\mathfrak{a} \trianglelefteq R$ is an arbitrary ideal.

**Lemma 2.6.** *The open subsets* $\mathrm{D}(f) := \mathrm{Spec}(R) - \mathrm{V}((f))$*, for any* $f \in R$*, form an IS basis of* $\mathrm{Spec}(R)$*.*

*Proof.* Complements of closed sets these are clearly open, and we can further calculate, for $f, g \in R$ and $\mathfrak{a} \trianglelefteq R$ an ideal:

$$\begin{aligned}
&\bigcup_{f\in\mathfrak{a}} \mathrm{D}(f) = \mathrm{Spec}(R) - \bigcap_{f\in\mathfrak{a}} \mathrm{V}((f)) = \mathrm{Spec}(R) - \mathrm{V}\left(\bigcup_{f\in\mathfrak{a}}(f)\right) = \mathrm{Spec}(R) - \mathrm{V}(\mathfrak{a}) \quad ; \\
&\mathrm{D}(f\cdot g) = \mathrm{Spec}(R) - \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} \ni fg\} = \\
&\qquad = \mathrm{Spec}(R) - \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} \ni f \vee \mathfrak{p} \ni g\} = \mathrm{D}(f) \cap \mathrm{D}(g) \quad .
\end{aligned}$$

The first assertion shows that every open subset can be covered by the $\mathrm{D}(f)$, and the second one that the basis is indeed intersection-stable. ☐

**Definition 2.7.** A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf $\mathcal{O}_X$ of commmutative rings on it (called the *structure sheaf*), such that at every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a *local* ring, i.e. a ring that has only one maximal ideal.

A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a sheaf morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ or equivalently $f^\flat : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, such that this morphism induces local ring maps on stalks, i.e. carries the (unique) maximal ideal into the maximal ideal.

**Proposition 2.8.** *We can make* $\mathrm{Spec}(R)$ *into a locally ringed space by defining the following structure sheaf* $\mathcal{O}_{\mathrm{Spec}(R)}$ *on it:*

$$\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{D}(f)) := R_f \tag{16}$$

*Transition maps are immediately given via the morphisms constructed in 2.2.*

*Proof.* By 2.4, it is enough to define a sheaf on a basis like this. We leave verifying the sheaf property as an easy exercise to the reader; further we note that the stalk at $\mathfrak{p} \in \mathrm{Spec}(R)$ is a local ring by writing:

$$\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} = \bigcup_{\mathfrak{p}\in \mathrm{D}(f)} \mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = \bigcup_{f\notin\mathfrak{p}} R_f = R_{\mathfrak{p}} \tag{17}$$

□

**Definition 2.9.** We define the category Aff of affine schemes with objects being the locally ringed spaces $\mathrm{Spec}(R)$, for $R$ any commutative ring, and morphisms of locally ringed spaces.

**Theorem 2.10.** *The functor* Spec *induces an equivalence of categories between* $\mathrm{CRing}^{op}$ *and* Aff*, with inverse given by taking global sections of the structure sheaf.*

*Proof.* Standard, but as we would first need to make Spec into a functor and the calculation of $\mathrm{Spec}(\mathcal{O}_X(X)) \cong X$ for $X \in \mathrm{Aff}$ is rather tedious, we refer to the vast literature. □

**Definition 2.11.** For $\mathrm{Spec}(R)$ any affine scheme and $M$ an $R$-module, we define the *quasicoherent sheaf* $\tilde{M}$ *associated to* $M$ as a sheaf on $\mathrm{Spec}(R)$ defined by

$$\tilde{M}(\mathrm{D}(f)) := M_f \tag{18}$$

with obvious transition maps. Again, the category of quasicoherent sheafs on $\mathrm{Spec}(R)$, denoted $\mathrm{QCoh}(\mathrm{Spec}(R))$, is equivalent via this construction and taking global sections to the category of modules on $R$.

With these definitions in mind, we can finally explain what an actual scheme is.

**Definition 2.12.** A *scheme* $(X, \mathcal{O}_X)$ is a locally ringed space such that for every $x \in X$, there is an open neighbourhood $x \in U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. We denote the category of schemes and morphisms of locally ringed spaces by Sch.

**Definition 2.13.** Let $X$ be a scheme, then a *quasicoherent sheaf* $\mathcal{F} \in \mathrm{QCoh}(X)$ is a sheaf such that on any affine open subset $U \subset X$ (it suffices to show this on an affine open cover), $\mathcal{F}|_U$ is a quasicoherent sheaf on the affine scheme $(U, \mathcal{O}_X|_U)$.

## 2.2 Superschemes

We will try to perform a similar construction after replacing rings with superrings.

**Definition 2.14.** Let $A$ be a superring (in particular, supercommutative). We then define a topological space $\mathbb{S}\mathrm{pec}(A)$ to be just $\mathrm{Spec}(A_0)$, namely the spectrum of the even part of $A$, considered as an ordinary commutative ring. Further, we define a structure sheaf on the IS basis $(\mathrm{D}(f) | f \in A_0)$ by

$$\mathcal{O}_{\mathbb{S}\mathrm{pec}\,A}(\mathrm{D}(f)) := A_f \tag{19}$$

*Remark.* This might seem a bit ad hoc, however since $A_1$ consists of nilpotent elements only, $\mathrm{D}(f) = 0$ for $f \in A_1$ anyway (see also 2.2), and even $\operatorname{Spec} A_0 = \operatorname{Spec} A$ since every prime ideal must contain the nil radical. Thus, we could just replace $A_0$ by $A$ at any point in this definition and obtain the same construction.

**Definition 2.15.** A *locally superringed space* $(S, \mathcal{O}_S)$ is defined analogously to a locally ringed space, the only difference is that the structure sheaf now takes values in superrings.

**Definition 2.16.** An *affine superscheme* is a locally superringed space of the form $\mathbb{S}\mathrm{pec}(A)$ for $A$ a superring - the proof that this is indeed locally superringed works just like for ordinary schemes. We denote the full subcategory of locally superringed spaces on these by SAff.

**Definition 2.17.** A *superscheme* is a locally superringed space $(S, \mathcal{O}_S)$ that, analogously to usual schemes, locally looks like an affine superscheme. We denote the category of such by $\mathbb{S}$Sch.

**Definition 2.18.** For $S = (S, \mathcal{O}_S)$ a superringed space, its *even part*, or bosonic quotient, $S_{even}$ is the ordinary ringed space $(S, (\mathcal{O}_S)_0)$.

**Theorem 2.19.** *The following conditions on a locally superringed space $(S, \mathcal{O}_S)$ are equivalent:*

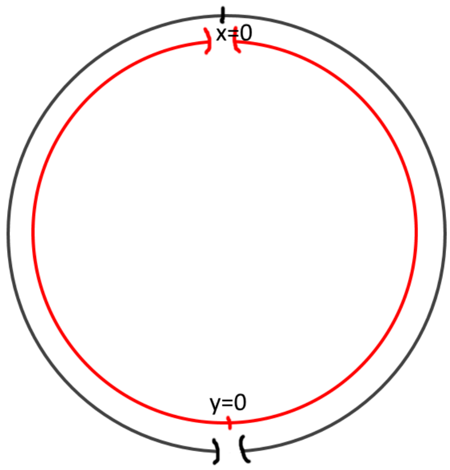
- $(S, \mathcal{O}_S)$ *is a superscheme*
- $S_{even} = (S, (\mathcal{O}_S)_0)$ *is a scheme, and* $(\mathcal{O}_S)_1$ *is a quasicoherent sheaf on this scheme.*

*Proof.* Since all of these statements can be checked on an affine open cover, we can reduce wlog. to the affine case.
$\underline{\Rightarrow}$ For an affine superscheme $S = \mathbb{S}\mathrm{pec}(A)$, we see immediately that $S_{even} = \operatorname{Spec} A_0$, thus an affine scheme. Since $A_1$ is an $A_0$-module we also follow that $(\mathcal{O}_S)_1 = \tilde{A}_1$ is indeed quasicoherent.
$\underline{\Leftarrow}$ If $(\mathcal{O}_S)_1$ is a quasicoherent module over the affine scheme $S = \operatorname{Spec} A_0$, then the equivalence in 2.11 shows that the $A_0$-module $A_1 := (\mathcal{O}_S)_1(S)$ satisfies $(\mathcal{O}_S)_1 = \tilde{A}_1$. Further, $\mathcal{O}_S(S) = A_0 \oplus A_1 =: A$ by definition of a locally superringed space is a superring, and we claim $X = \mathbb{S}\mathrm{pec}\, A$ finishing our proof. This holds trivially as topological spaces, and on affine opens $\mathrm{D}(f)$ as well since $A_f = (A_0)_f \oplus (A_1)_f$. $\square$

This characterization is often easier to check and can without much effort be generalized to define $\mathbb{Z}$-graded superschemes or dg schemes.

**Example 2.20.** *For $k$ any field, we can define the affine superspace* $\mathbb{S}\mathrm{pec}(\mathrm{Sym}^* k^{p|q}) = \mathbb{S}\mathrm{pec}(k[x_1, \ldots, x_p, \theta_1, \ldots, \theta_q]) = \mathbb{A}^{p|q}$*; for $k = \mathbb{R}$ this should be imagined as an algebraic analogon to $\mathbb{R}^{p|q}$.*

**Example 2.21.** *We define projective superspace* $\mathbb{P}_k^{p|q}$ *by gluing together* $p+1$ *copies of* $\mathbb{A}_k^{p|q}$*, just like for usual projective space. On the right, we sketch this for* $p=1$*, where two copies* $\mathbb{A}_k^{1|q} = \mathbb{S}\mathrm{pec}\, k[\frac{x}{y}, \frac{\theta_1}{y}, \dots, \frac{\theta_q}{y}]$ *and* $\mathbb{A}_k^{1|q} = \mathbb{S}\mathrm{pec}\, k[\frac{y}{x}, \frac{\theta_1}{x}, \dots, \frac{\theta_q}{x}]$ *are glued along* $\mathbb{S}\mathrm{pec}\, k[(\frac{x}{y})^{\pm 1}, \frac{\theta_1}{y}, \dots, \frac{\theta_1}{y}]$ *in the obvious way.*




*There are of course efficient ways to write this down, in particular the* Proj*-Construction (see [BRP20]) of a* $\mathbb{Z}$*-*$\mathbb{Z}_2$*-bigraded superalgebra. Note that the last expression below exhibits the structure sheaf as the exterior algebra on a locally free sheaf on the bosonic space* $\mathbb{P}_k^p$*, we therefore say that* $\mathbb{P}_k^{p|q}$ *is split:*

$$\mathbb{P}_k^{p|q} = \bigcup_{i=0}^{p} \mathbb{A}_k^{p|q} = \mathbb{P}\mathrm{roj}(\mathrm{Sym}^*\, k^{p|q}) = \left(\mathbb{P}_k^p, \bigwedge^* \mathcal{O}_{\mathbb{P}_k^p}(-1)^{\oplus n}\right) \tag{20}$$

**Proposition 2.22.** *A scheme* $(X, \mathcal{O}_X)$ *is uniquely determined by:*

- *Its functor of points* $\mathrm{Hom}_{\mathrm{Sch}}(-, X) : \mathrm{Sch}^{op} \to \mathrm{Set}$
- *The restriction of this functor to* $\mathrm{Hom}_{\mathrm{Sch}}(-, X) : \mathrm{Aff}^{op} \to \mathrm{Set}$
- *The functor corepresented by it* $\mathrm{Hom}_{\mathrm{Sch}}(X, -) : \mathrm{Sch} \to \mathrm{Set}$

*In particular, we have an embedding* $\mathrm{Sch} \hookrightarrow \mathcal{PSh}(\mathrm{Aff}) \simeq \mathcal{PSh}(\mathrm{CRing}^{op})$.

*Proof.* The first and third claim are just the (opposite) Yoneda Lemma; and the second one is proved exactly analogously to (9), replacing charts by affine open subsets and atlases by affine coverings. □

*Remark.* For $X \in \mathrm{Sch}$, also denote its functor of points by $X(-)$, and again call the elements of $X(S)$, for any $S \in \mathrm{Sch}$, *$S$-points* of $X$.

The similarity of this statement and its proof to $\mathrm{Mfd} \hookrightarrow \mathcal{PSh}(\mathrm{Cart})$ is no accident, as we will see later - and it should come as no surprise that it holds verbatim for superschemes:

$$\mathbb{S}\mathrm{Sch} \hookrightarrow \mathcal{PSh}(\mathrm{SAff}) \simeq \mathcal{PSh}(\mathrm{SRing}^{op}) \tag{21}$$

The plan of these statements is again to define new spaces by explicitly giving their functors of points and showing that those are representable; further, properties like allowing a Lie group or algebraic group structure can often be detected a lot easier on the functors of points.

# 3 Applications of the Functor of Points

Remember the following constructions for superrings, that we can actually promote to functors:

- Given a commutative ring $R$, we can construct a superring $i(R)$ by setting $i(R)_0 = R$ and $i(R)_1 = 0$ with the induced multiplication - this extends to a functor $\mathrm{CRing} \hookrightarrow \mathrm{SRing}$.
- Given a superring $A$, we can just forget about its odd part and obtain a commutative ring $A_0$ with well defined multiplication, as $A_0 \cdot A_0 \subseteq A_0$ in our grading. We denote this by a functor $(-)_0 : \mathrm{SRing} \to \mathrm{CRing}$
- Finally, again with $A$ a superring, we can forget about the grading to obtain a non-commutative ring, and divide this ring by the two-sided ideal $A_1^2 \oplus A_1$ generated by (formerly) odd elements $A_1$. This gives a commutative ring as is easy to see, denote this by a functor $(-)/(-)_1 : \mathrm{SRing} \to \mathrm{CRing}$

**Proposition 3.1.** *We have natural equivalences between the following morphism sets, for $R$ a commutative ring and $A$ a superring:*

$$\mathrm{Hom}_{\mathrm{CRing}}(A/A_1, R) \cong \mathrm{Hom}_{\mathrm{SRing}}\left(A, i(R)\right) \quad , \tag{22}$$

$$\mathrm{Hom}_{\mathrm{SRing}}(i(R), A) \cong \mathrm{Hom}_{\mathrm{CRing}}(R, A_0) \quad . \tag{23}$$

*Proof.* A superring morphism $A \to i(R)$ must send $A_1$ to $i(R)_1 = 0$ and therefore consists equivalently of a usual ring morphism $A/A_1 \to R$ by the homomorphism theorem - the other direction follows by the same argument.

A superring morphism $i(R) \to A$ consists of usual ring morphisms $i(R)_0 = R \to A_0$ and $i(R)_1 = 0 \to A_1$, the latter being trivial data. □

We say that the functor $(-)/(-)_1$ is left adjoint to $i$ and the functor $(-)_0$ is right adjoint to $i$, and denote this situation by the following diagram:

$$\begin{tikzcd} \mathrm{CRing} \arrow[r, hook, "i" description] & \mathrm{SRing} \arrow[l, bend right, "(-)/(-)_1"'] \arrow[l, bend left, "(-)_0"] \end{tikzcd}$$

Since the theory of (super-)schemes is nothing but a geometrization of the theory of (super-)rings, namely just a subcategory of the presheaf category on CRing respectively SRing as we have seen, we expect there to be a way to extend these constructions to this richer setting. It is possible to formalize this (via a composition of a Left Kan Extension and Sheafification on the (Super-)Zariski-Topos), but we will opt to only give the results:

**Proposition 3.2.** *We obtain three pairwise adjoint functors between schemes and superschemes that fit in the following diagram:*

$$\begin{tikzcd} \mathrm{Sch} \arrow[r, hook, "i"] & \mathbb{S}\mathrm{Sch} \arrow[l, bend right, "(-)_{even}"'] \arrow[l, bend left, "(-)_{bos}"] \end{tikzcd}$$

- *The even part of a superscheme was already defined in 2.18*
- *The natural inclusion of schemes into superschemes just sends a scheme $(X, \mathcal{O}_X)$ to itself, equipped with the trivial quasi-coherent sheaf describing the odd part - it is easy to see that this functor is again fully faithful.*
- *The bosonic reduction of a superscheme $(S, \mathcal{O}_S)$ (distinguish this from the usual reduction of a scheme) is formed by taking the sheaf quotient of the structure sheaf by its odd part, namely by applying the functor $(-)/(-)_1$ from above on every open set and then sheafifying.*

*Proof.* We let the reader verify that the above functors are indeed well-defined, and only show the adjunction properties, namely the natural equivalences of morphism sets. Since all of our functors don't change the underlying topological spaces, it will suffice to show that, for a fixed continuous map $f : S \to X$, the sheaf morphisms satisfy the required property - again, the reader should verify that being a local ring map on stalks is also preserved by the following calculation.

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{Sh}(S_{even},\mathrm{CRing})}\left(f^{-1}\mathcal{O}_X, \mathcal{O}_{S_{even}}\right) = \mathrm{Hom}_{\mathcal{Sh}(S,\mathrm{CRing})}\left(f^{-1}\mathcal{O}_X, (\mathcal{O}_S)_0\right) = \\ &\quad \mathrm{Hom}_{\mathcal{Sh}(S,\mathrm{SRing})}\left(f^{-1}\mathcal{O}_{i(X)}, \mathcal{O}_S\right) \end{aligned} \tag{24}$$

Check yourself that pointwise applying $i$ to $f^{-1}\mathcal{O}_X$ obtains $f^{-1}\mathcal{O}_{i(X)}$, similarly for $f_*$. This proves the first adjunction (note that morphisms of locally (super-)ringed space induce sheaf morphisms in the *other* direction!), and the second one follows from

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{Sh}(S,\mathrm{CRing})}\left(\mathcal{O}_S, f_*\mathcal{O}_{i(X)}\right) = \mathrm{Hom}_{\mathcal{PSh}(S,\mathrm{CRing})}\left(\mathcal{O}_S, f_*\mathcal{O}_{i(X)}\right) = \\ &\quad \mathrm{Hom}_{\mathcal{PSh}(S,\mathrm{CRing})}\left({}^{\mathcal{O}_S(-)}\!\big/_{(\mathcal{O}_S(-))_1}, f_*\mathcal{O}_X\right) = \mathrm{Hom}_{\mathcal{Sh}(X,\mathrm{CRing})}\left(\mathcal{O}_{S_{even}}, f_*\mathcal{O}_X\right) \end{aligned} \tag{25}$$

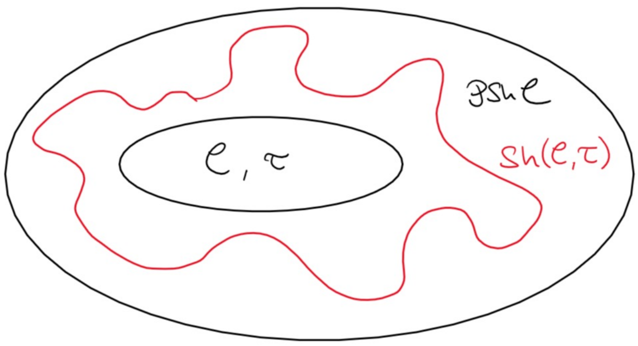
where we note $(S, \mathcal{O}_S)_{even} = \left(S, \left({}^{\mathcal{O}_S}\!\big/_{(\mathcal{O}_S)_1}\right)^{sh}\right)$ and use the universal property of sheafification. □

*Remark.* The way we have discovered $(-)_{even}$ and $(-)_{bos}$, retrospectively, was by defining how their functors of points (resp. their opposites) should look like and then giving explicit constructions to show that they are indeed representable. If one wanted to further refine this strategy, one could develop explicit representability criteria as it done for example in [CCF11].

Finally, as we have swept of lot of technical details under the rug in this presentation, let us take a look at how to further formalize many of the previous constructions. We had seen that for any category $\mathcal{C}$, the presheaf category $\mathcal{PSh}(\mathcal{C})$ can be constructed from $\mathcal{C}$ be freely gluing together objects along arbitrary morphisms between them - unfortunately, general objects in this new category are usually way too outlandish and general to be useful.

Often however, certain subcategories of the presheaf category are of interest. Namely, you might have heard of the *idempotent completion* or *Karoubi envelope* of a category, which in a way forms a thin hull around $\mathcal{C}$ in the presheaf category. Also, number theorists are probably familiar with *Ind- and Pro-completions* (e.g. as in *profinite groups*); the former and variants thereof can also be found as a full subcategory of $\mathcal{PSh}(\mathcal{C})$.

For us, the main interest is to somehow specify a way in which model objects of $\mathcal{C}$ should be glued, and only regard presheaves that occur by such a well-behaved gluing. This works by equipping $\mathcal{C}$ with a construction plan, a so-called *Grothendieck Topology* $\tau$, that lets us (in a similar manner as on topological spaces) define a subcategory $\mathcal{Sh}(\mathcal{C}, \tau) \subseteq \mathcal{PSh}(\mathcal{C})$, the so-called *sheaf topos* on the *site* $(\mathcal{C}, \tau)$.




While this still doesn't give us exactly what we want, objects in these topoi are usually much better behaved than arbitrary presheaves, and very close to being geometric spaces. We have had glimpses on:

- The topos of *smooth sets* that contains the category of smooth manifolds - here, we only allow gluing along open subsets of Cartesian spaces
- The topos of *supersmooth sets* that contains the category of supermanifolds
- The *big Zariski topos* that contains schemes
- The *big Super-Zariski topos* that contains superschemes

These constructions truly inherit the functor-of-points philosophy, and we refer the reader to [nLa21] and the other chapters of this book project for further information and more abstract nonesense.

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