Supermanifolds and the Batchelor's Theorem

Seminar: Supergeometry

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1 Supermanifolds: Algebro-Geometric Approach

We will still start with the formal definition and explain it step by step:

Definition 1.1. *A* ***supermanifold*** *of dimension $(p|q)$ is a manifold M of dimension p, together with a sheaf of superalgebras A. These superalgebras should be locally isomorphic to the space $\underline{\Lambda R^q} := C^{\infty}(\mathbb{R}^p) \quad \Lambda^{\bullet}(\xi_1, ..., \xi_q)$, meaning the exterior (grassmann) algebra of q-generators.*

At first this does not seem like a intuitive definition. So we first start by recalling definitions:

Definition 1.2. *A* ***$\mathbb{Z}_2$-graded ring*** *is a ring, s.t. the ring is decomposed into a direct sum*

$$R = R_0 + R_1$$

of additive groups, such that $R_m R_n \quad R_{m+n}$ modulo 2.

Definition 1.3. *An algebra is a* ***$\mathbb{Z}_2$-graded algebra*** *if the underlying ring is a $\mathbb{Z}_2$-graded ring. It is called a superalgebra. Also for the commutator the following relation holds: $a \quad A_i, b \quad A_j \quad ab = (-1)^{ij} ba$*

These definitions were already given. Now we need the definition of a sheaf:

Definition 1.4. *A Sheaf [4] is a functor F from the category of open sets on a topological space X to the category of groups/rings/algebras, s.t.*

1. *$U \quad_{op} X \quad$ a set $F(U)$. These sets are called sections.*
2. *For every open sets U, V s.t. $V \quad U \quad$ a restriction homomorphism $res_{V,U} : F(U) \to F(V)$, s.t.*
   a) *$res_{U,U} : F(U) \to F(U)$ is the identity*
   b) *for $W \quad V \quad U$: $res_{W,U} = res_{W,V} \circ res_{V,U}$*

   Then F is called a presheaf. If additionally
3. *For an open covering $\{U_i\}_{i \quad I}$ of U and $s, t \quad F(U)$, such that $s|_{U_i} = t|_{U_i} \quad i \quad I \quad s = t$ (Uniqueness)*
4. *If for an open covering $\{U_i\}_{i \quad I}$ of U there exists an $s_i \quad F(U_i)$, s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \quad I, (i = j) \quad s \quad F(U)$, s.t. $s|_{U_i} = s_i$ for all $i \quad I$ (Gluing)*

are fullfilled, then F is called a sheaf.

An example for sheaves is:

Example 1.1. *Let M be a smooth manifold, then it also is a topological space, and $C^{\infty}(M)$ be the set of smooth functions on M. Then F is a sheaf, with $F(U)$ for U $_{op}$ M sections being $C^{\infty}(U)$. For U, V $_{op}$ M with V U, then*

$$res_{V,U} : C^{\infty}(U) \to C^{\infty}(V)$$
$$f \to f|_V$$

For f $C^{\infty}(U)$ and W V U open

$$f|_U = f$$

and

$$f|_V|_W = f|_W$$

hold.
Therefore F is a presheaf. A well-defined function is completely defined by its values on an open covering and therefore uniqueness is fulfilled. Gluing is also easy to see, since if two functions agree on an open subset i.e. the intersection, then it can be continued on the open subsets. Therefore one can glue di erent di erentiable functions together if they agree on all intersections of an open covering.
F is a sheaf.

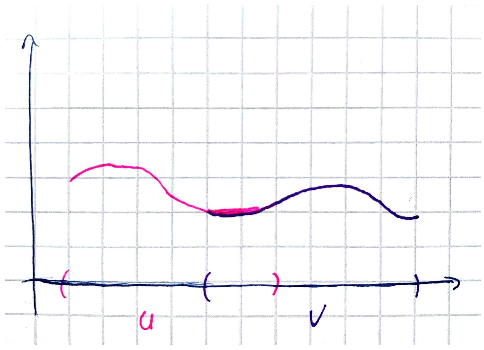

Figure 1: Gluing of functions on R

With this definition clear, we continue with the definition of the exterior algebra:

Definition 1.5. *Given a vector space V over a field K, with $dim(V) = q$. Let*

$$T^{\bullet}(V) = \bigoplus_{k \geq 0} V^{k}$$

be a tensor space and a Z graded algebra. Let $J(V)$ be the homogeneous ideal generated by

$$< v \quad w + w \quad v | v, w \quad V >$$

Then

$$\Lambda^{\bullet}(V) = T^{\bullet}(V)/J(V)$$

is a Z_2-graded-commutative and the ***exterior algebra****.*

Example 1.2. *An easy example for an exterior algebra are di erential forms. This will be explained later.*

Here one begins to see why a supermanifold is actually called manifold, since we have the chart from A to $C^\infty(M) \quad \Lambda(R^q)$ This example is actually used on supermanifolds. One can understand those graded algebras as splitted into two parts, the first one are the even numbered rings and the second part are the odd numbered rings. This makes sense since the even numbered grassmann forms commute, while the odd anticommute.

2 Supermanifolds: Concrete approach

There is a second maybe more intuitive approach to supermanifolds. It is comparable to the definition of smooth manifolds:

Definition 2.1. *A topological space M is a **supermanifold** of dimension $(p|q)$ if there exists a complete atlas mapping to the superspace $R^{(p|q)}$.*

This atlas has to fulfill the same requirements as the atlas for a common smooth manifold:

Definition 2.2. *Let M be a topological space. A complete Atlas A is a collection of tupels (U_i, φ_i), s. t.:*

1. $U_i \quad {}_{op} M \quad i \quad I$
2. $\varphi_i : U_i \to V \quad {}_{op} R^{(p|q)}$ *is a di eomorphism*
3. $M = \bigcup_{i \quad I} U_i$

Now one may have already seen a problem. We did not define smoothness as a concept on superspaces yet and also one would have to take a look at topology on grassmann numbers.
Since all of this could take some time, we will skip this approach. The equivalence of the two definitions is not easy to see, but is explained here [2], and also smoothness and topology on this spaces is still an active topic of discussion. The elegant way to handle this is to use the algebro-geometric definition, even if it seems a little less intuitive. If you want to learn more about this topic chapter 4 of [4] explains the smoothness and topology of this definition quite well.

3 Examples

We will talk about some examples of supermanifolds:

Example 3.1. *Obviously a superspace of dimension $(p|q)$ is a trivial example of a supermanifold.*

Example 3.2. *Another example is the* $(m|0)$ *dimensional supermanifold constructed on* $(M, C^\infty(M))$. *Here the sheaves of smooth functions on the m dimensional manifold M, are the supermanifold.*

Definition 3.1. *If* E *is vector bundle of a manifold* M *and*

$$\pi : E \to M$$

is the projection of to vector bundle onto the manifold, then a section of E *is a continuous map*

$$\sigma : M \to E$$

s.t. $\pi(\sigma(x)) = x \quad x \quad M$. *The sections on* E *are denoted by* $\Gamma(E)$.

Example 3.3. *All other not trivial examples can be put in the form* $(M, \Gamma(\Lambda(E)))$, *with E smooth vector bundle over manifold* M. *The dimension of this supermanifold is then* $(dim(M), dim(E))$. *The name split manifold comes from the splitting in an exact sequence of sheaves:*

$$0 \to P_2 \to A \to C^\infty \quad E \to 0$$

Example 3.4. *Let* M *be a manifold of dimension m (m even). Then* $T\ M$ *is the co-tangential bundle and a vector bundle of dimension* n.
Now take a look at the the exterior product of the $T\ M$ *i.e. the set of differential forms on* M:

$$\Lambda(T\ M)$$

From there we can take a look at the sections of differential forms:

$$\Gamma(\Lambda(T\ M))$$

We now take sheaves from M *to* $C^\infty(M) \quad \Lambda(T\ M)$. *We can find open neighborhoods* U_i, *s.t. the sheaves can locally isomorphic to*

$$C^\infty(U_i) \quad \Lambda(\mathrm{R}^m)$$

We can then identify the superalgebra by:

$$A_0 = C^\infty(M) \quad \Lambda^2(T\ M) \quad ... \quad \Lambda^m(T\ M)$$

$$A_1 = \Lambda^1(T\ M) \quad ... \quad \Lambda^{m-1}(T\ M)$$

And we find local isomorphisms:

$$A_0|_{U_i} \to C^\infty(M) \quad \Lambda^2(\mathrm{R}^m) \quad ... \quad \Lambda^m(\mathrm{R}^m)$$

$$A_1|_{U_i} \to \Lambda^1(\mathrm{R}^m) \quad ... \quad \Lambda^{m-1}(\mathrm{R}^m)$$

To look at T_pM *at different points, we need a change of chart for the co-tangent space. Since the tangent space induces the co-tangent space, it is enough to transform the tangent space. One finds, that the tangent space is changed via* $Gl(m)$ *with* $C^\infty(M)$*-functions as entries(as well as all other m-dimensional vector bundles). But* $T\ M$ *induces the sheaf of differential forms, which is why* $Gl(m)$ *also represents the automorphisms between differential forms.*

4 Batchelor's Theorem

The previous example leads to the main theorem of this talk:

4.1 The Statement

The Main Theorem 1. *If E is a real vector bundle over a manifold M, let $\Lambda(E)$ be the associated exterior bundle and let $\Gamma(\Lambda(E))$ be the sheaf of sections of $\Lambda(E)$. Then every supermanifold over M is isomorphic to $\Gamma(\Lambda(E))$ for some vector bundle E over X.[1]*

Remark 1. *This statement only holds for a smooth, real manifold M, C^∞, R, but this will be topic of a later talk.*

4.2 Explanation

While the statement, especially if one is not that familiar with sheaves, also seems confusing in the beginning, the idea is as useful, as it is surprising.
As already mentioned earlier in the direct approach to supermanifolds problems arise concerning smoothness and topological properties of the superspace. Here it becomes more clear, why the algebro-geometric approach is superior. We know exactly how vector bundles of a manifold transform and how stuff acts on them.
E.g. the transformation of a vector bundle E on different points of the underlying manifold M, is given by $Gl(q)$ $\quad dim(E) = q$. This way we reduced a not at all trivial or rigorous problem of a supermanifold to a simple problem we already know well. Vector bundles and sheaves of exterior algebras on them.

4.3 Cohomology on Sheaves

We will start by introducing some cohomology, in particular the Čech-Cohomology. The Čech-Cohomology is a rather handy definition of cohomology on sheaves, but is equivalent to the abstract definition for separable Hausdorff spaces, e.g. manifolds.

Let M be a manifold and F a presheaf of abelian groups on M. Let U be an open cover of M.

Definition 4.1. *A **q-simplex** σ is a collection of $q+1$ ordered sets chosen from U, s.t the intersection of all those sets is not empty. The intersection of those sets is then called the **support** and is denoted $|\sigma|$.*

Definition 4.2. *Now let $\sigma = \{U_i\}_{i \in \{0,1,\dots,q\}}$ be such a simplex. Then the **j-th partial boundary** of σ is defined to be the $(q-1)$-simplex*

$$\partial_j \sigma = \{U_i\}_{i \in \{0,1,..,q\}/\{j\}}$$

*The **boundary** is defined as:*

$$\partial \sigma = \sum_{j=0}^{q} (-1)^{j+1} \partial_j \sigma$$

This can be viewed as an element of the free abelian group spanned by the simplices of U.

Definition 4.3. *A **q-cochain** of U with coe cients in F is a map which associates with each element σ a map $F(|\sigma|)$ and we denote the set of all q-cochains of U with coe cients in F by $C^q(U, F)$.*

Definition 4.4. *We can make this into a cochain-complex with $(C^\bullet(U, F)$ by defining*

$$\delta_q : C^q(U, F) \to C^{q+1}(U, F)$$

$$\delta_q f(\sigma) = \sum_{j=0}^{q+1} (-1)^j res_{|\sigma|}^{|\partial_j \sigma|} f(\partial_j \sigma)$$

One can show, that $\delta_{q+1} \circ \delta_q = 0$

Definition 4.5. *The **q-cocycles** are $Z_q(U, F) = ker(\delta_q) \quad C^q(U, F)$.*

Definition 4.6. *The **q-cobounds** are $B_q(U, F) = im(\delta_{q-1}) \quad C^q(U, F)$*

Definition 4.7. *Since $\delta_{q+1} \circ \delta_q = 0$, it makes sense to define*

$$\check{H}^q(U, F) := H^q((C^\bullet(U, F), \delta)) := Z^q(U, F)/B^q(U, F)$$

Now one chooses refinement and defines:

$$\check{H}^q(F) = lim_{U\to} \check{H}^q(U, F)$$

Here we only discussed abelian cohomology. Non-abelian cohomology gets even more complicated and results in having only the $\check{H}^1(F)$-class and it being a pointed set.

4.4 Sketch of the Proof

Since the proof of the theorem gets complex very fast, we will just sketch the idea behind the proof. If you are interested in a more detailed proof, you can find it in Batchelor's original paper [1].

We make use of the local triviality of sheaves on supermanifolds and then we can use results from cohomology.

We denote by

$$Aut(\underline{\Lambda R^q}) = \{\text{Automorphisms of the sheaf of } Z_2\text{-graded algebras } \underline{\Lambda R^q}|_U\}$$

Recall that vector bundles are locally trivial fibrations in R^q such that the transition functions are

$$U_i \cap U_j \overset{C^\infty}{\to} Gl_q(R)$$

Meaning $Gl_q(R)$ is the structure group. Therefore we can identify:

$$\left\{\begin{matrix}\text{vector bundles on } M\\ \text{of dimension } k\end{matrix}\right\} / isomorphisms \longleftrightarrow \check{H}^1(M, \underline{Gl_q})$$

A standard result from cohomology is:

Theorem 4.1. *There is a bijection between the set $H^1(M, Aut(\underline{\Lambda R^q}))$ and the set of isomorphism classes of supermanifolds over M of odd-dimension q.*

The proof can be found in [3], p.44.
This theorem already proves useful, since if we can find a bijection to the first cohomology class of a vector bundle, then we find that the isomorphism classes are the same. This is exactly what we will do.
We define two sheaves, which will prove useful later:

Definition 4.8.

$$C(U) = \underline{Der}(U) \quad \Lambda R^q(e,2)$$

Where the $\Lambda R^q(e,r) = \square_{2i \geq r} \Lambda^{2i} R$ and $\underline{Der}(U)$ is the sheaf of derivations on $C^\infty(U)$(i.e. the set of sections of the tangent bundle of U).

$$D(U) = \underline{GL(q)(U)} \times Hom(R^q, \Lambda R^q(o,3))$$

Here $\underline{Gl(q)(U)}$ is the sheaf of invertible $q \times q$ matrices with entries in C^∞ and $\Lambda R^q(o,3) = \square_{2i+1 \geq r} \Lambda^{2i+1} R$.

This way we get another theorem, whichs proof is very technical and can also be found in [1].

Theorem 4.2. *Let P be the sheaf given by*

$$P(U) = C(U) \times D(U)$$

. Then there is an isomorphism of sheaves of sets:

$$\Phi : P \to Aut(\underline{\Lambda R^q})$$

Now one can define a filtration on P.
Define:

Definition 4.9.

$$P_i(U) = \{\alpha \quad Aut(\underline{\Lambda R^q})(U) | \alpha(x) - x \quad \underline{\Lambda R^q}(i)(U) \quad x \quad \underline{\Lambda R^q})(U)\}$$

With $\underline{\Lambda R^q}(i)(U) = \square_{j \geq i} \underline{\Lambda^i R}$

We can see from the definition, that

$$id = P_{q+1} \leq P_q \leq ... \leq P_0 = Aut(\underline{\Lambda R^q})$$

This is because the definition only gets stricter for bigger i.
Another theorem states:

Theorem 4.3. *Identifying P with $Aut(\underline{\Lambda R^q})$ via Φ for every $U \quad_{op} M$, we get:*

1. *P_i is a normal subgroup of P for $i \quad \{0, ..., q+1\}$*

2. $P_i(U) \cong \begin{cases} \underline{Der}(U) \otimes \Lambda R^q(e,i) \times Hom(R^q, \underline{\Lambda R^q(o,i)}) & \text{for } i \geq 2 \\ P(U) & \text{for } i = 0,1 \end{cases}$

3. $P_i/P_{i+1}(U) \cong \begin{cases} \underline{Der}(U) \otimes \Lambda^i R^q & \text{for } i \geq 2, \text{ even} \\ Hom(R^q, C^\infty(U) \otimes \Lambda^i R^q) & \text{for } i \geq 3, \text{ odd} \\ \underline{Gl(q)}(U) & \text{for } i = 1 \end{cases}$

4. *From 2., we can define an action of $C^\infty(U)$ on P_i for $i \in \{0, ..., q\}$, which coincides with the standard action of $C^\infty(U)$ on the quotients $P_i/P_{i+1}(U)$. Moreover, if z is an automorphism of $P_i/P_{i+1}(U)$ arising from a conjugation by an element in $P(U)$, then z is an automorphism of a $C^\infty(U)$ module.*

From that we get a restatement of the main theorem, which connects all the theorems we used:

Theorem 4.4. *The map of sheaves*

$$p : P \to P/P_2 \cong \underline{GL(q)}$$

induces an isomorphism on cohomology

$$p_* : H^1(M, P) \to H^1(M, P/P_2) = H^1(M, \underline{Gl(q)})$$

Whats now left to proof is the bijectivity of p_*. Since this is too much for the length of this talk and it will not contribute to deeper understanding, the rest can be found here [1].

Remark 2. *The idea behind this reformulation is comparing the group of automorphisms on a supermanifold as a sheaf of groups on M (2). While obviously $P_1(U) = P(U) = Aut(\underline{\Lambda R^q})$ are the group of automorphisms on a supermanifold.*