Def.: (Graded Ring)

Let R be a unitary ring and Γ be an abelian group.
We call R a Γ-graded ring if $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ with each R_γ an abelian subgroup of R and $R_{\gamma_1} R_{\gamma_2} \subseteq R_{\gamma_1 + \gamma_2}$ $\forall \gamma_1, \gamma_2 \in \Gamma$.

Terminology:

- $R^h := \bigcup_{\gamma \in \Gamma} R_\gamma$ the set of all homogeneous elements
- $r \in R_\gamma$, $r \neq 0$ is a called a homogeneous element of degree $\deg(r) = \gamma$

$\Rightarrow$ $\deg : R^h \setminus \{0\} \to \Gamma$; $r \mapsto \deg(r)$

$\Rightarrow$ $\deg(r_1 r_2) = \deg(r_1) + \deg(r_2)$ $\forall r_1, r_2 \in R^h \setminus \{0\}$

Most important: $\mathbb{Z}$-grades rings and $\mathbb{Z}/2\mathbb{Z}$-graded rings.

A $\mathbb{Z}/2\mathbb{Z}$-graded ring is also called a superring

Theorem: Let R be a Γ-graded ring. Then we have:

(i) $1 \in R_0$ (ii) R_0 is subring of R (iii) R_γ is R_0-bimodule (iv) $r \in R_\gamma$ unit $\Rightarrow r^{-1} \in R_{-\gamma}$

Proof:

(i) $1 \in R_0 \Rightarrow 1 = \sum_{\gamma \in \Gamma} r_\gamma$. $s \in R_\delta \rightsquigarrow s = s \cdot 1 = \sum_{\gamma \in \Gamma} \underbrace{s r_\gamma}_{\in R_{\gamma+\delta}} \Rightarrow s r_\gamma = 0 \quad \forall \gamma \neq 0$

$\Rightarrow \forall s \in R: \ s r_\gamma = 0 \quad \forall \gamma \neq 0 \Rightarrow 1 r_\gamma = r_\gamma = 0 \quad \forall \gamma \neq 0 \Rightarrow 1 = r_0 \in R_0$

(ii) R_0 is subgroup of R, $1 \in R_0$ and $R_0 R_0 \subseteq R_0$

(iii) R_γ is subgroup of R and $R_0 R_\gamma R_0 \subseteq R_\gamma$

(iv) $r \in R_\gamma$ unit, $r^{-1} = \sum_\delta r_\delta \Rightarrow 1 = r r^{-1} = \sum_\delta \underbrace{r s_\delta}_{\in R_{\gamma+\delta}} \Rightarrow r s_\delta = s_\delta = 0 \ \forall \gamma+\delta \neq 0$

$\Rightarrow r^{-1} = s_{-\gamma} \in R_{-\gamma}$

Remark: If R is a $\mathbb{Z}$-graded domain we aditionally have $R^\times \subseteq R^h$.

Proof: $a = \sum_\gamma a_\gamma \in R$ define $w(a) = \max_{\gamma \in \mathbb{Z}} \{a_\gamma \neq 0\} - \min_{\gamma \in \mathbb{Z}} \{a_\gamma \neq 0\} \geq 0$. For a, b we have

$$ab = (\underbrace{a_{\gamma_{min}}}_{\neq 0} + \dots + \underbrace{a_{\gamma_{max}}}_{\neq 0})(\underbrace{b_{\delta_{min}}}_{\neq 0} + \dots + \underbrace{b_{\delta_{max}}}_{\neq 0}) = \underbrace{a_{\gamma_{min}} b_{\delta_{min}}}_{\neq 0 \ \in R_{\gamma_{min}+\delta_{min}}} + \dots + \underbrace{a_{\gamma_{max}} b_{\delta_{max}}}_{\neq 0 \ \in R_{\gamma_{max}+\delta_{max}}}$$

$\Rightarrow w(ab) = w(a) + w(b) \Rightarrow 0 = w(1) = w(ab) = w(a) + w(b)$

$\Rightarrow w(a) = 0$ and $w(b) = 0 \Rightarrow a, b \in R^h$.

We also see: k $\mathbb{Z}$-graded Field $\Rightarrow k_0 = k$ and $k_d = \{0\} \ \forall d \neq 0$

Examples:

- R abitrary ring $\Rightarrow$ R is Γ-graded ring with $R_0 = R$ and $R_\gamma = \{0\}$ $\forall \gamma \neq 0$
  $\rightsquigarrow$ trivial grading

- V Vector space $\Rightarrow$ $T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$, $S(V) = \bigoplus_{n \in \mathbb{N}} S^n(V)$, $\Lambda(V) = \bigoplus_{n \in \mathbb{N}} \Lambda^n(V)$
  are $\mathbb{Z}$-graded rings.

- R comm. Ring $\Rightarrow$ $R[x_1, \dots, x_n]$ is $\mathbb{Z}$-graded ring with
  $$R[x_1, \dots, x_n]_d = \{ f \in R[x_1, \dots, x_n] \mid f(\lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_1, \dots, x_n) \ \forall (x_1, \dots, x_n) \in R^n \}$$

Def.: (Subgroup and Quotient Grading)

R Γ-graded ring and $\Omega \leq \Gamma$ subgroup. Then:

- $R_\Omega := \bigoplus_{\gamma \in \Omega} R_\gamma$ is Ω-graded ring
- $R = \bigoplus_{\alpha + \Omega \in \Gamma/\Omega} R_{\alpha+\Omega}$ with $R_{\alpha+\Omega} := \bigoplus_{\omega \in \Omega} R_{\alpha+\omega}$ is Γ/Ω-graded ring.

Remark:

Because we have the canonical projection $\pi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ we can see every $\mathbb{Z}$-graded ring as a $\mathbb{Z}/2\mathbb{Z}$-graded ring, i.e. superring

$\Rightarrow$ $T(V)$, $S(V)$, $\Lambda(V)$ are superrings with e.g.

$$T(V) = T(V)_{\bar 0} \oplus T(V)_{\bar 1} \; ; \; T(V)_{\bar 0} = \bigoplus_{d \in \mathbb{N}} V^{\otimes 2d} \; , \; T(V)_{\bar 1} = \bigoplus_{d \in \mathbb{N}} V^{\otimes 2d+1}$$

Def.: (Graded Ringhom.)

R, S Γ-graded rings. $f: R \to S$ is a graded ringhom. if we have

f is ringhom. and preserves the grading, i.e. $f(R_\gamma) \subseteq S_\gamma \quad \forall \gamma \in \Gamma$

Def.: (Graded-Commutative Ring)

R Γ-graded ring and $\varepsilon: \Gamma \to \mathbb{Z}/2\mathbb{Z}$ a group hom.

R is a (Γ, ε)-graded-commutative ring if we have

$$ab = (-1)^{|a||b|} ba \quad \forall a, b \in R^h \setminus \{0\} \quad \text{with } |\cdot| = (\varepsilon \circ \deg)(\cdot)$$

Remark:

- If $\varepsilon : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ is the trivial grouphom., then R is graded-comm.
  $\Leftrightarrow$ R is graded and R is commutative
- For $\mathbb{Z}$- and $\mathbb{Z}/2\mathbb{Z}$-graded rings we usually take
  $\varepsilon : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ the canonical projection and $\varepsilon : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ the identity.
  $\Rightarrow$ With this $\mathbb{Z}$-graded-comm. and $\mathbb{Z}/2\mathbb{Z}$-grad.-comm. coincide

Example:

- $T(V)$ is neither commutative nor graded-commutative
- $S(V)$ is commutative but not graded-commutative
- $\Lambda(V)$ is not commutative but graded-commutative

Def.: (Supercommutator and Supercenter)

- Supercommutator: $[a,b] := ab - (-1)^{|a||b|} ba$
- Supercenter: $Z(A) = \{ a \in A \mid \forall b \in A \ [a,b] = 0 \}$

If $[a,b] = 0$ we say a supercommutes with b.

We have: A grad.-comm. $\Leftrightarrow A = Z(A)$

The supercommutator fulfills:

- $[a,b] = ab - (-1)^{|a||b|} ba = -(-1)^{|a||b|} (ba - (-1)^{|a||b|} ab) = -(-1)^{|a||b|} [b,a]$
- $[a,[b,c]] + (-1)^{|c|(|a|+|b|)} [c,[a,b]] + (-1)^{|a|(|b|+|c|)} [b,[c,a]] = 0$

Remark: If A is a supercomm. Ring then we have

$a^2 = \frac{1}{2}[a,a] = 0$ for all $a \in A_{\bar{1}}$ $\Rightarrow$ All elements of degree $\bar{1}$ are nilpotent.

Def.: (Tensorproduct of Graded Rings)

R, S (Γ, ε)-graded rings. Then $R \otimes_{\mathbb{Z}} S$ is again a (Γ, ε)-graded ring with

$$(R \otimes_{\mathbb{Z}} S)_\gamma := \bigoplus_{\substack{\gamma_1, \gamma_2 \in \Gamma \\ \gamma_1 + \gamma_2 = \gamma}} R_{\gamma_1} \otimes_{\mathbb{Z}} R_{\gamma_2} .$$

The ring multiplication on $R \otimes_{\mathbb{Z}} S$ is given by:

$$(r_1 \otimes s_1)\cdot(r_2 \otimes s_2) = (-1)^{|s_1||r_2|} r_1 r_2 \otimes s_1 s_2 \quad \text{for } r_1, r_2 \in R^h \setminus \{0\},\ s_1, s_2 \in S^h \setminus \{0\}$$

We have $(R \otimes_{\mathbb{Z}} S)_{\gamma_1} (R \otimes_{\mathbb{Z}} S)_{\gamma_2} \subseteq (R \otimes_{\mathbb{Z}} S)_{\gamma_1 + \gamma_2}$

because

$$\deg((r_1 \otimes s_1)\cdot(r_2 \otimes s_2)) = \deg((-1)^{|s_1||r_2|} r_1 r_2 \otimes s_1 s_2) = \deg(r_1 r_2) + \deg(s_1 s_2)$$

$$= \deg(r_1) + \deg(r_2) + \deg(s_1) + \deg(s_2)$$

$$= \deg(r_1 \otimes s_1) + \deg(r_2 \otimes s_2)$$

The factor $(-1)^{|r_2||s_1|}$ is needed because we want that when R, S grad.-comm. then also $R \otimes_{\mathbb{Z}} S$ grad.-com.

- $(r_1 \otimes s_1)(r_2 \otimes s_2) = (-1)^{|s_1||r_2|} r_1 r_2 \otimes s_1 s_2 = (-1)^{|s_1||r_2| + |r_1||r_2| + |s_1||s_2|} r_2 r_1 \otimes s_2 s_1$

$$= \underbrace{(-1)^{|s_1||r_2| + |r_1||r_2| + |s_1||s_2| + |r_1||s_2|}}\quad r_2 \otimes s_2 \cdot r_1 \otimes s_1$$

$$= (-1)^{(|r_1|+|r_2|)(|s_1|+|s_2|)}$$

$$= (-1)^{|r_1 \otimes r_2||s_1 \otimes s_2|}$$

- R, R', S, S' Γ-graded rings and $f: R \to R'$, $g: S \to S'$ graded ringhom., then $f \otimes g: R \otimes_{\mathbb{Z}} S \to R' \otimes_{\mathbb{Z}} S'$; $r \otimes s \mapsto f(r) \otimes g(s)$ is again a graded ringhom.

We have: $r \otimes s \in R \otimes_{\mathbb{Z}} S_\gamma$ with $r \in R_{\gamma_1}$, $s \in S_{\gamma_2}$, $\gamma_1 + \gamma_2 = \gamma$

$$\Rightarrow f \otimes g\,(r \otimes s) = f(r) \otimes g(s) \in (R' \otimes_{\mathbb{Z}} S')_\gamma \text{ because } f(r) \in R'_{\gamma_1},\ g(s) \in S'_{\gamma_2}$$

- $f\otimes g(r_1\otimes s_1 \cdot r_2\otimes s_2) = f\otimes g((-1)^{|s_1||r_2|} r_1 r_2 \otimes s_1 s_2) = (-1)^{|s_1||r_2|} f(r_1 r_2)\otimes g(s_1 s_2)$

$$= (-1)^{|s_1||r_2|} f(r_1) f(r_2) \otimes g(s_1) g(s_2)$$

$$= \underbrace{(-1)^{|s_1||r_2|} (-1)^{|f(r_2)||g(s_1)|}}_{=1} f(r_1)\otimes g(s_1) \cdot f(r_2)\otimes g(s_2)$$

$$= (f\otimes g)(r_1\otimes s_1) \cdot f\otimes g(r_2\otimes s_2)$$

Def.: (Graded Module)

R Γ-graded ring and M a right module over R.

M Γ-graded right module $:\Longleftrightarrow$ • $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ with M_γ abelian subgroup of M

• $M_{\gamma_1} R_{\gamma_2} \subseteq M_{\gamma_1+\gamma_2}$ $\forall\, \gamma_1, \gamma_2 \in \Gamma$

Analogous to graded rings: • $M^h := \bigcup_{\gamma \in \Gamma} M_\gamma$

• $\deg : M^h \setminus \{0\} \to \Gamma;\ m \mapsto \deg(m) = \gamma \Leftrightarrow m \in M_\gamma$

• $\deg(mr) = \deg(m) + \deg(r)$ $\forall\, m \in M^h \setminus \{0\}, r \in R^h \setminus \{0\}$

Remark: The definitions for Γ-graded left module and Γ-graded bimodule are analogous. Again

- $\mathbb{Z}/2\mathbb{Z}$-graded module over a $\mathbb{Z}/2\mathbb{Z}$-graded ring
  $\Longleftrightarrow$ Supermodule over a superring
- $\mathbb{Z}/2\mathbb{Z}$-graded module over a trivially graded field
  $\Longleftrightarrow$ Supervectorspace

- If R is a supercommutative ring and M a right supermodule over R then we have a left multiplication on M such that M becomes a superbimodule. It is given by: $r.m := (-1)^{|r||m|} m.r \quad \forall r \in R^h \setminus \{0\}, m \in M^h \setminus \{0\}$

Proof:

- $(rs)\cdot m = (-1)^{|rs||m|} m\cdot(rs) = (-1)^{(|r|+|s|)|m|} (-1)^{|s||r|} m\cdot(sr) = (-1)^{|r||m|+|s||m|+|s||r|} (m\cdot s)\cdot r$

$= (-1)^{|r||m|+|r||s|} (sm)\cdot r = (-1)^{|r|\underbrace{(|m|+|s|)}_{=|sm|}} (sm)r = r\cdot(s\cdot m)$

- $(r\cdot m)\cdot s = (-1)^{|r||m|} (m\cdot r)\cdot s = (-1)^{|r||m|+|r||s|} (m\cdot s)\cdot r = r\cdot(m\cdot s)$

- If R, S are superrings and $f: R \to S$ a superring hom. with $f(R) \subseteq Z(S)$, then S becomes an superbimodule over R with

$r_1 \cdot s := f(r_1)\cdot s$ and $s\cdot r_2 = s\cdot f(r_2)$

Proof.:

- $r \cdot s = (-1)^{|r||s|} s \cdot r = (-1)^{|r||s|} s \cdot f(r) = f(r) \cdot s$
- $s \cdot r = (-1)^{|r||s|} r \cdot s = (-1)^{|r||s|} f(r) \cdot s = s\, f(r)$

Example:

- Every Superring can be seen as a supermodule over itself
- $T(V)$, $S(V)$, $\Lambda(V)$ can be seen as supervectorspaces over the groundfield k of the ordinary vectorspace V.

From now on we are only interested in Γ-graded modules over Γ-graded-comm. rings.

Def.: (Graded Module hom.)

M, N Γ-graded right modules over Γ-graded ring R.

- $f: M \to N$ is a graded right module hom. if f is a right module hom and $f(M_\gamma) \subseteq N_\gamma$ $\forall \gamma \in \Gamma$.

- $f: M \to N$ is a graded right module hom. of degree δ if f is right module hom. and $f(M_\gamma) \subseteq N_{\gamma+\delta}$ $\forall \gamma \in \Gamma$.

Notation:

- $\mathrm{Hom}(M,N) = \mathrm{Hom}(M,N)_0$ set of all graded modulehom. between M and N
- $\mathrm{Hom}(M,N)_\gamma$ set of all graded modulehom. of degree γ between M and N

Def.: (Shifting Functor)

M Γ-graded module. For each $\delta \in \Gamma$ we define the Γ-graded module $M(\delta)$ with $M(\delta)_\gamma := M_{\gamma+\delta}$ and for $f: M \to N$ graded modulehom. $f(\delta): M(\delta) \to N(\delta)$ with $f(\delta)(m) = f(m)$

$f(\delta)$ is again a graded modulehom. because

$m \in M(\delta)_\gamma = M_{\gamma+\delta} \Rightarrow f(\delta)(m) = f(m) \in N_{\gamma+\delta} = N(\delta)_\gamma$

We also have $M(\delta_1)(\delta_2) = M(\delta_1+\delta_2)$ and the natural bijections

$$\mathrm{Hom}(M,N)_\delta \cong \mathrm{Hom}(M(-\delta),N) \cong \mathrm{Hom}(M, N(\delta))$$

Remark.:

In the case of supermodules over a superring we write

$\Pi M = M(\bar{1})$, $\Pi f = f(\bar{1})$ and call Π the parity change functor.

Here we have $\Pi\Pi M = M$.

- For the set of all $f: M \to N$ module hom. (not necessarily graded) between two graded modules we write $\underline{\mathrm{Hom}}(M,N)$, the internal hom.

- $\underline{\mathrm{Hom}}(M,N)$ important because when Γ finite then $\underline{\mathrm{Hom}}(M,N)$ is again a Γ-graded module in contrast to $\mathrm{Hom}(M,N) = \mathrm{Hom}(M,N)_0$

Reason:

$f \in \underline{\mathrm{Hom}}(M,N)$. We define $f_\delta(m) := \sum_\gamma f(m_{\gamma-\delta})_\gamma$

Let $m \in M_\alpha$ $m \neq 0$. $\Rightarrow$ $f_\delta(m_\alpha) = \sum_\gamma \delta_{\alpha,\gamma-\delta} f(m_{\gamma-\delta})_\gamma = f(m_\alpha)_{\alpha+\delta} \in N_{\alpha+\delta}$

$\Rightarrow f_\delta \in \mathrm{Hom}(M, N)_\delta$

We also have $f(m) = \sum_{\gamma, \delta} f(m_{\gamma-\delta})_\gamma = \sum_\delta f_\delta(m)$

$$\Rightarrow \underline{\mathrm{Hom}}(M, N) \cong \bigoplus_{\delta \in \Gamma} \mathrm{Hom}(M, N)_\delta$$

Therefore every general right modulehom. between M and N can be written as a unique sum of graded right modulehom. of degree δ.

We can define compatible left and right multiplication on $\underline{\mathrm{Hom}}(M, N)$ via

$$(r_1 f)(m) := r_1 f(m) \quad \text{and} \quad (f r_2)(m) = f(r_2 m)$$

Proof.:

Left structure: $(rf)(m_1+m_2) = rf(m_1+m_2) = rf(m_1) + rf(m_2) = (rf)(m_1) + (rf)(m_2)$

$(rf)(mr') = rf(mr') = rf(m)r' = (rf)(m)r'$

$((r_1r_2)f)(m) = (r_1r_2)f(m) = r_1(r_2f(m)) = (r_1(r_2f))(m)$

Right structure: $(fr)(m_1+m_2) = f(r(m_1+m_2)) = f(rm_1) + f(rm_2) = (fr)(m_1) + (fr)(m_2)$

$(fr)(mr') = f(rmr') = f(rm)r' = (fr)(m)r'$

$(f(r_1r_2))(m) = f((r_1r_2)m) = f(r_1(r_2m)) = ((fr_1)r_2)(m)$

They are compatible because:

- $(fr)(m) = f(rm) = (-1)^{|r||m|}f(mr) = (-1)$

They are compatible because:

- $$(fr)(m) = f(rm) = (-1)^{|r||m|} f(mr) = (-1)^{|r||m|} f(m)r$$
$$= (-1)^{|r||m| + |r||f(m)|} r f(m) = (-1)^{|r|(|m|+|f(m)|)} r f(m)$$
$$= (-1)^{|r||f|} (rf)(m)$$

- $$(rf)(m) = r f(m) = (-1)^{|r||f(m)|} f(m) r = (-1)^{|r||f(m)|} f(mr) = (-1)^{|r|(|f(m)|+|m|)} f(rm)$$
$$= (-1)^{|r||f|} (fr)(m)$$

From this we also see:

$f(mr) = f(m) r$ but $f(rm) = (-1)^{|f||r|} r f(m)$

Def.: (Direct Sum + Tensorproduct)

Let M, N be Γ-graded modules ove a Γ-graded-comm. ring R.

Then $M \oplus N$ with $(M \oplus N)_\gamma := M_\gamma \oplus N_\gamma \;\; \forall \gamma \in \Gamma$ and

$M \otimes_R N$ with $(M \otimes_R N)_\gamma := \bigoplus\limits_{\substack{\gamma_1, \gamma_2 \in \Gamma \\ \gamma_1 + \gamma_2 = \gamma}} M_{\gamma_1} \otimes M_{\gamma_2}$ are again

Γ-graded modules.

Def.: (Graded Algebra over a Graded-comm. Ring)

Let M be a Γ-graded module over a Γ-graded-comm. ring R.

We call M an graded R-algebra if M is additionally a Γ-graded ring.
If M is a Γ-graded-comm. ring then we call M a graded-com. R-alg..

Alterativeely:

M Γ-graded ring. We call M an graded R-alg. if we have a graded ring hom. $f: A \to M$ with $f(A) \subseteq Z(M)$.

- Again the tensor product $M \otimes_R N$ of two graded R-algebras is a graded R-algebra by defining $m_1 \otimes n_1 \cdot m_2 \otimes n_2 = (-1)^{|n_1||m_2|} m_1 m_2 \otimes n_1 n_2$

Def.: (Graded Algebrahom.)

Let M, N be two graded R-algebras and $f: M \to N$ a graded module hom.
We call f an graded R-algebrahom. if $f(m_1 m_2) = f(m_1) f(m_2)$ for all $m_1, m_2 \in M$.

Alternatively:

The Diagram

$$\begin{tikzcd} & M \arrow[d, "f"] \\ R \arrow[ur, "g_M"] \arrow[r, "g_N"'] & N \end{tikzcd}$$

commutes

i.e.
- $f(rm) = f(g_M(r)m) = f(g_M(r)) f(m) = g_N(r) f(m)$
$= r f(m)$
- $f(mr) = f(m g_M(r)) = f(m) f(g_M(r)) = f(m) g_N(r)$
$= f(m) r$

Def.: (Free Module)

Let M be a supermodule over a supercommutative ring R.

We call M a free module if M has a basis of homogeneous elements.

If this basis is finite then we call M a free module of finite rank

and write $\operatorname{srank} M = (p|q)$ with p and q the number of

even / odd basis elements.

Example: Let A be a supercomm. ring.

- $A^{p|q} := \underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{\Pi A \oplus \cdots \oplus \Pi A}_{q}$ is a free module of finite rank $p|q$.

  A Basis of $A^{p|q}$ is given by $(\underbrace{e_1, \dots, e_p}_{\text{even}}, \underbrace{e_{p+1}, \dots, e_{p+q}}_{\text{odd}})$

  with $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \dots$.

- The rank of a free module over the supercomm. ring A is uniquely determined.

  To see this we take $J_A = \langle A_{\bar{1}} \rangle = A_{\bar{1}} \oplus A_{\bar{1}}^2$ the irrelevant ideal in A and build the quotient $A/J_A \cong \frac{A_{\bar{0}}}{A_{\bar{1}}^2}$. We have $1 \notin J_A$

$\Rightarrow A/J_A$ is a comm. ring with unit $\Rightarrow \exists$ field F and ringhom. $A/J_A \to F$

$\Rightarrow A^{p|q} \otimes_A F$ is supervectorspace of rank $p|q$ $\rightsquigarrow$ $p|q$ uniquely determined.

Vector - Matrix - Representation

Let M be a free supermodule of rank $p|q$ over the supercomm. ring A.

Then we have an even isomorphism $f: M \to A^{p|q}$ that sends

the homogeneous basis of M to the homogeneous basis $(e_1, \dots, e_{p+q})$ of $A^{p|q}$.

Every $a \in A^{p|q}$ can now be written as

$$a = \sum_{i=1}^{p+q} e_i a^i \iff a \leftrightarrow \begin{pmatrix} a^1 \\ \vdots \\ a^p \\ a^{p+1} \\ \vdots \\ a^{p+q} \end{pmatrix}$$; colum vector representation

But we also have $a = \sum_{i=1}^{p+q} e_i a^i = \sum_{i=1}^{p+q} (-1)^{|e_i||a^i|} a^i e_i$

$\Leftrightarrow \quad a \leftrightarrow \left(a^1 \dots a^p \; (-1)^{|a^{p+1}|} a^{p+1} \dots (-1)^{|a^{p+q}|} a^{p+q} \right)$; row vector representation

$= \left(a^1 \dots a^p \; \tilde{a}^{p+1} \dots \tilde{a}^{p+q} \right)$ with $\tilde{a}^i = (-1)^{|a^i|} a^i$

We have
- $a \in (A^{p|q})_{\bar{0}} \Leftrightarrow a_1, \dots, a_p$ even & $a_{p+1}, \dots, a_{p+q}$ odd

$$\Leftrightarrow a = \begin{pmatrix} \vdots \\ \text{even} \\ \vdots \\ \text{odd} \\ \vdots \end{pmatrix} \begin{matrix} \}\, p \\ \\ \}\, q \end{matrix}$$

- $a \in (A^{p|q})_{\bar{1}} \Leftrightarrow a_1, \dots, a_p$ odd & $a_{p+1}, \dots, a_{p+q}$ even

$$\Leftrightarrow a = \begin{pmatrix} \vdots \\ \text{odd} \\ \vdots \\ \text{even} \\ \vdots \end{pmatrix} \begin{matrix} \}\, p \\ \\ \}\, q \end{matrix}$$

- $f: A^{p|q} \to A^{r|s}$ a module hom.

- $x \in A^{p|q}$ with $x = \sum_{i=1}^{p+q} e_i x^i$; $f(x) = y \in A^{r|s}$ with $y = \sum_{j=1}^{r+s} e_i' y^i$

We now have $f(e_i) = \sum_{j=1}^{r+s} e_j T^j_i$

With this $y = f(x) = \sum_{j=1}^{r+s} \sum_{i=1}^{p+q} e_j' \underbrace{T^j_i x^i}_{= y^i}$

$$\Rightarrow \begin{pmatrix} y^1 \\ \vdots \\ y^{r+s} \end{pmatrix} = \underbrace{\begin{pmatrix} T_1^1 & \cdots & T_1^{p+q} \\ \vdots & & \vdots \\ T_{r+s}^1 & \cdots & T_{r+q}^{p+q} \end{pmatrix}}_{= \begin{array}{c} \left(\begin{array}{c|c} T_{\bar 0}^{\bar 0} & T_{\bar 0}^{\bar 1} \\ \hline T_{\bar 1}^{\bar 0} & T_{\bar 1}^{\bar 1} \end{array}\right) \begin{array}{l} \}\, r \\ \}\, s \end{array} \\ \underbrace{\quad}_{p} \ \underbrace{\quad}_{q} \end{array}} \begin{pmatrix} x^1 \\ \vdots \\ x^{p+q} \end{pmatrix}$$

We have • $f \in \mathrm{Hom}(A^{p|q}, A^{r|s})_{\bar{0}} \iff \begin{pmatrix} T^{\bar{0}}_{\bar{0}} & T^{\bar{0}}_{\bar{1}} \\ T^{\bar{1}}_{\bar{0}} & T^{\bar{1}}_{\bar{1}} \end{pmatrix} = \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}$

• $f \in \mathrm{Hom}(A^{p|q}, A^{r|s})_{\bar{1}} \iff \begin{pmatrix} T^{\bar{0}}_{\bar{0}} & T^{\bar{0}}_{\bar{1}} \\ T^{\bar{1}}_{\bar{0}} & T^{\bar{1}}_{\bar{1}} \end{pmatrix} = \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}$

• Scalar multiplication :

$$(af)(x) = a\, f(x) = a \sum_{j=1}^{r+s} \sum_{i=1}^{p+q} e_j' \, T^j_{\ i} \, x^i = \sum_{j=1}^{r+s} \sum_{i=1}^{p+q} e_j' \, (-1)^{|a||e_j'|} \, T^j_{\ i} \, x^i$$

$$\Rightarrow \quad a \begin{pmatrix} T^{\bar{0}}_{\bar{0}} & T^{\bar{0}}_{\bar{1}} \\ T^{\bar{1}}_{\bar{0}} & T^{\bar{1}}_{\bar{1}} \end{pmatrix} = \begin{pmatrix} a\,T^{\bar{0}}_{\bar{0}} & a\,T^{\bar{0}}_{\bar{1}} \\ \tilde{a}\,T^{\bar{1}}_{\bar{0}} & \tilde{a}\,T^{\bar{1}}_{\bar{1}} \end{pmatrix} \quad \text{with } \tilde{a} = (-1)^{|a|} a$$

$$(fa)(x) = f(ax) \rightsquigarrow ax = a\sum_{i=1}^{p+q} e_i x^i = \sum_{i=1}^{p+q} e_i \, (-1)^{|a||e_i|} a x^i$$

$$\Rightarrow \quad f(ax) = \sum_{j=1}^{r+s} \sum_{i=1}^{p+q} e_j' T^j_i \, (-1)^{|a||e_i|} a x^i$$

$$\Leftrightarrow \quad \begin{pmatrix} T^{\bar 0}_{\bar 0} & T^{\bar 0}_{\bar 1} \\ T^{\bar 1}_{\bar 0} & T^{\bar 1}_{\bar 1} \end{pmatrix} \cdot a = \begin{pmatrix} T^{\bar 0}_{\bar 0} a & T^{\bar 0}_{\bar 1} \tilde a \\ T^{\bar 1}_{\bar 0} a & T^{\bar 1}_{\bar 1} \tilde a \end{pmatrix}$$

- **Parity transpose:**

$A^{p|q}$ has rank $(p|q)$ and basis $(\underbrace{e_1, \dots, e_p}_{\text{even}}, \underbrace{e_{p+1}, \dots, e_{p+q}}_{\text{odd}})$

$\rightsquigarrow$ $\Pi A^{p|q}$ has rank $(q|p)$ and the basis $(\underbrace{e_1, \dots, e_p}_{\text{odd}}, \underbrace{e_{p+1}, \dots, e_{p+q}}_{\text{even}}) = (\underbrace{e_{p+1}, \dots, e_{p+q}}_{\text{even}}, \underbrace{e_1, \dots, e_p}_{\text{odd}})$

$\Rightarrow$ $x \in A^{p|q}$ $x = \sum_{i=1}^{p+q} e_i x^i$

$$x \longleftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ x_{p+1} \\ \vdots \\ x_{p+q} \end{pmatrix} \longleftrightarrow (x_1 \dots x_p \; \tilde{x}_{p+1} \dots \tilde{x}_{p+q})$$

is represented in $\Pi A^{p|q}$ as

$$\Pi x \longleftrightarrow \begin{pmatrix} x^{p+1} \\ \vdots \\ x^{p+q} \\ x^1 \\ \vdots \\ x^p \end{pmatrix} \longleftrightarrow (x^{p+1} \dots x^{p+q} \; \tilde{x}^1 \dots \tilde{x}^p)$$

We have $\Pi f(x) = f(x)$

$$\Rightarrow \begin{pmatrix} T^{\bar{0}}_{\bar{0}} & T^{\bar{0}}_{\bar{1}} \\ T^{\bar{1}}_{\bar{0}} & T^{\bar{1}}_{\bar{1}} \end{pmatrix}^{\Pi} = \begin{pmatrix} T^{\bar{1}}_{\bar{1}} & T^{\bar{1}}_{\bar{0}} \\ T^{\bar{0}}_{\bar{1}} & T^{\bar{0}}_{\bar{0}} \end{pmatrix}$$

We have: $(A+B)^{\Pi} = A^{\Pi} + B^{\Pi}$, $(A \cdot B)^{\Pi} = A^{\Pi} \cdot B^{\Pi}$, $(A^{\Pi})^{\Pi} = A$

◦ Supertranspose

Let A be a supercom. ring and M be a free supermodule over A of finit rank.

Then we have the dual module $M^* := \underline{\mathrm{Hom}}(M, A)$.

Analogously to ordinary linear algebra we have a evaluation map

$$\langle \cdot , \cdot \rangle : M^* \otimes_A M \to A \; ; \; w \otimes m \mapsto \langle w, m \rangle = w(m)$$

For $f : M \to N$ we get now a map $f^* : N^* \to M^*$

with the property $\underbrace{\langle f^*(w), m \rangle}_{= f^*(w)(m)} = (-1)^{|f||w|} \langle w, f(m) \rangle = (-1)^{|f||w|} w(f(m))$

Let $A^{p|q}$ with basis $(e_1, \dots, e_{p+q})$

Then for $(A^{p|q})^* = \underline{\mathrm{Hom}}(A^{p|q}, A)$ we have the dual basis $(e^1, \dots, e^{p+q})$ with $e^i(e_j) = \delta^i_j$

$\Rightarrow$ Every $w \in (A^{p|q})^*$ with $w(e_i) = w_i$ can be represented as

$$w = \sum_{i=1}^{p+q} w_i e^i \quad \Leftarrow \quad w \longleftrightarrow (w_1 \dots w_{p+q}) \quad \text{row vector representation}$$

$$w \longleftrightarrow \begin{pmatrix} w_1 \\ \vdots \\ w_p \\ \tilde{w}_{p+1} \\ \vdots \\ \tilde{w}_{p+q} \end{pmatrix} \quad \text{colum vector representation}$$

$\Rightarrow$ w in row rep. and m in colum rep. then $\langle w, m\rangle = w(m)$ is just matrix multiplication: $w(x) \leftrightarrow (w_1 \ \dots \ w_{p+q}) \begin{pmatrix} x^1 \\ \vdots \\ x^{p+q} \end{pmatrix}$

We now have: $\langle f^*(w), x\rangle = (-1)^{|f||w|} w(f(x))$

$$= (-1)^{|f||w|} \sum_{j=1}^{r+s} \sum_{i=1}^{p+q} w_j T^j_i x^i = (-1)^{|f||w|} \sum_{j=1}^{r+s} \sum_{i=1}^{p+q} (-1)^{|T^j_i||w_j|} T^j_i w_j x^i$$

From this one can see that $f^*(w)$ represented as a colum vector is given by

$$\begin{pmatrix} T^{\bar 0}_{\bar 0} & T^{\bar 0}_{\bar 1} \\ T^{\bar 1}_{\bar 0} & T^{\bar 1}_{\bar 1} \end{pmatrix}^{st} \begin{pmatrix} w^1 \\ \vdots \\ w^{p+q} \end{pmatrix} \text{ with } \begin{pmatrix} T^{\bar 0}_{\bar 0} & T^{\bar 0}_{\bar 1} \\ T^{\bar 1}_{\bar 0} & T^{\bar 1}_{\bar 1} \end{pmatrix}^{st} = \begin{pmatrix} {T^{\bar 0}_{\bar 0}}^t & (-1)^{|f|} {T^{\bar 1}_{\bar 0}}^t \\ -(-1)^{|f|} {T^{\bar 0}_{\bar 1}}^t & {T^{\bar 1}_{\bar 1}}^t \end{pmatrix}$$

Especially we have:

$$\left(\begin{pmatrix} T^{\bar{0}}{}_{\bar{0}} & T^{\bar{0}}{}_{\bar{1}} \\ T^{\bar{1}}{}_{\bar{0}} & T^{\bar{1}}{}_{\bar{1}} \end{pmatrix}^{st}\right)^{st} = \begin{pmatrix} T^{\bar{0}}{}_{\bar{0}} & -T^{\bar{0}}{}_{\bar{1}} \\ -T^{\bar{1}}{}_{\bar{0}} & T^{\bar{1}}{}_{\bar{1}} \end{pmatrix} \neq \begin{pmatrix} T^{\bar{0}}{}_{\bar{0}} & T^{\bar{0}}{}_{\bar{1}} \\ T^{\bar{1}}{}_{\bar{0}} & T^{\bar{1}}{}_{\bar{1}} \end{pmatrix}$$

The supertranspose fullfills:

- $(A+B)^{st} = A^{st} + B^{st}$
- $(AB)^{st} = (-1)^{|A||B|} B^{st} A^{st}$
- $(A)^{st\,st\,st\,st} = A \;,\; A^{st\,st} \neq A \,,\; (A^{\Pi})^{st\,\Pi} = A$

- Supertrace :

We have $\underline{End}(M) = \underline{Hom}(M,M) \cong M^* \otimes M$

for M with finite rank by sending

$f \in \underline{End}(M) \mapsto \tilde{f} = \sum_j (-1)^{|e_j|(1+|f|)} e^j \otimes f(e_j) \in M^* \otimes M$

and $\sum_i s_i^* \otimes s_i \in M^* \otimes M \mapsto \underbrace{(\sum_i s_i^* \otimes s_i)}_{\in \underline{End}(M)}(t) = \sum_i (-1)^{|s_i||t|} s_i^*(t) s_i$

But we also have the evaluation map: $\langle \cdot , \cdot \rangle : M^* \otimes M \to A$

$\langle \sum_i s_i^* \otimes s_i \rangle = \sum_i s_i^*(s_i)$.

The supertrace of f is now given by

$$str(f) = \langle \hat{f} \rangle$$ like the trace in ordinary linear algebra.

Writing f as a matrix in standard coordinates yields:

$$str\begin{pmatrix} T^{\bar{0}}{}_{\bar{0}} & T^{\bar{0}}{}_{\bar{1}} \\ T^{\bar{1}}{}_{\bar{0}} & T^{\bar{1}}{}_{\bar{1}} \end{pmatrix} = tr(T^{\bar{0}}{}_{\bar{0}}) - (-1)^{|f|} tr(T^{\bar{1}}{}_{\bar{1}})$$

The supertrace has the following properties:

- By definition $str: \underline{End}(M) \to A$ is a superorphism of A-Modules
- $str(xy) = (-1)^{|x||y|} str(yx)$

- $\mathrm{Str}(XY) = (-1)^{|X||Y|}\, \mathrm{str}(YX)$

$$\Rightarrow \mathrm{Str}(XY) - (-1)^{|X||Y|}\,\mathrm{str}(YX) = \mathrm{str}(XY - (-1)^{|X||Y|}\,YX)$$
$$= \mathrm{Str}([X,Y]) = 0$$

$$\Rightarrow \mathrm{Str}(CXC^{-1}) = \underbrace{(-1)^{|C||XC^{-1}|}}_{\substack{=1 \\ \text{because } |C|=0}}\, \mathrm{str}(XC^{-1}C) = \mathrm{str}(X)$$

$\rightsquigarrow$ Alternative way to see that supertrace is basis independent.

- $\mathrm{Str}(X^{st}) = \mathrm{str}(X)$ and $\mathrm{str}(X^{T}) = -(-1)^{|X|}\,\mathrm{str}(X)$

- $\mathrm{id}_{p|q} \in \underline{\mathrm{End}}(A^{p|q})$ is represented as $\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & 1 & \ddots \end{pmatrix}$

$\rightsquigarrow \quad \mathrm{str}(\mathrm{id}_{p|q}) = p - q$