# Lie-Groups und Representation Theory

## Exercise Sheet 5

Summer Term 2019

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**Deadline:** 21.05.2019, 11:00 Uhr

**Exercise 5.1** *Relevance of the identity - 8 Points*

(a) Let $G_1$, $G_2$ be topological groups and let $f : G_1 \to G_2$ be a homomorphism which is continuous at the identity $e \in G_1$. Show that $f$ is continuous. *Hint*: A similar statement also holds for any other element $g \neq e \in G_1$, however the identity is usually most easily understood. Use the topological definition of continuity *at a point*[1].

(b) Let $G$ be a connected topological group and let $U \in G$ be a neighbourhood of the identity $e \in G$. For every $n \in \mathbb{N}$ we denote by $U^n$ the set of elements $u_1 \cdots u_n$, with $u_i \in U$. Show that

$$\bigcup_{n \in \mathbb{N}_0} U^n = G,$$

i.e. $U$ algebraically generates $G$. *Hint*: Assume without loss of generality $U = U^{-1}$ and remember that a topological space $X$ is connected if and only if the only clopen (= closed and open) subsets are $X$ and the empty set $\emptyset$.

**Exercise 5.2** *The center of topological groups - 8 Points*

Let $G$ be a topological Group.

(a) Show that if $G$ is Hausdorff, the center $Z(G)$ is closed in $G$.

(b) Show that if $G$ is path-connected, any discrete normal subgroup $H \triangleleft G$ is in the center $Z(G)$. *Hint*: Consider the map $G \to H$, $g \mapsto ghg^{-1}$ for a fixed element $h \in H$.

(c) *Bonus*: The fundamental group $\pi_1(G)$ of a connected Lie group $G$ acts on the universal cover $\widetilde{G}$ by deck transformations. By applying (b) to the kernel of the universal covering map $\widetilde{G} \to G$, show that $\pi_1(G)$ is commutative.

**Exercise 5.3** *Haar measure of a discrete group - 8 Points*

Let $G$ be a locally compact topological group with a left Haar measure $\mu$. Show that $G$ is discrete if and only if $\mu(\{e\}) > 0$, where $e \in G$ is the identity.

**Exercise 5.4** *"Essentially" a Lie group - 10 bonus points*

As we will learn soon, a Lie group is a group which is also a smooth manifold such that both structures are compatible in a certain sense. Not every topological group is a Lie group: The rational numbers $\mathbb{Q}$ equipped with the subspace topology of $\mathbb{R}$ are a counterexample. But clearly, every Lie group is a topological group, simply by forgetting the smooth structure. The resulting object is still a topological manifold which in particular is locally homeomorphic to Euclidian space. As it turns out, this local model is sufficient to conclude a converse statement, which was the content of Hilbert‘s (by now solved) fifth problem: A locally Euclidian topological group is isomorphic to a Lie group. In

[1] $f : X \to Y$ is continuous at $x \in X$ if for any given neighborhood $N$ of $f(x)$ there is a neighborhood $M$ of $x$ such that $f(M) \subseteq N$.

practice, verifying a topological group to be locally Euclidian might be very difficult, so one might hope to weaken the requirements for such a statement to be true. The goal of this exercise is to give an example which illustrates that *local compactness* is not enough.

Let $p$ be a prime. We define the *$p$-adic norm* $\|\cdot\|_p$ on the integers $\mathbb{Z}$ by defining $\|n\|_p := p^{-j}$, where $p^j$ is the largest power of $p$ that divides $n$ (setting $\|0\|_p := 0$). The resulting metric space can be completed in the usual way to obtain the *$p$-adic integers* $\mathbb{Z}_p$.

(a) Show that $(\mathbb{Z}_p, +, 0)$ is a topological group with respect to the topology induced by the $p$-adic norm. *Hint*: Use the triangle inequality. You will not need the strong triangle inequality below.

For a metric space, being compact is equivalent to being complete and totally bounded. $\mathbb{Z}_p$ is complete by construction, so in order to verify compactness it is enough to

(b) show that $\mathbb{Z}_p$ is totally bounded. *Hint*: To prove that $\mathbb{Z}_p$ can be covered by finitely many $\epsilon$-balls, it is enough to consider $p^{-n}$-balls. All of these balls are cosets of the $p^{-n}$-ball around zero and it is $\mathbb{Z}_p/p^n\mathbb{Z}_p \simeq \mathbb{Z}/p^n\mathbb{Z}$.
*Alternatively*, show that every sequence in $\mathbb{Z}_p$ contains a convergent subsequence.

Therefore the $p$-adic integers are also locally compact. Recall that locally Euclidian in particular implies locally connected. $\mathbb{Z}_p$ can be seen to be very far from locally connected as follows.

(c) Show that the metric induced by the $p$-adic norm is even an *ultrametric*: A metric $d : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{R}$ which satisfies the *strong* triangle inequality

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for $x, y, z \in \mathbb{Z}_p$. *Hint*: Consider elements in the representation $x = \pm p^m r$ and $y = \pm p^n s$.

(d) Show that an ultrametric space is totally disconnected, i.e. a space whose connected components are the one-point sets. *Hint*: If you can prove that in an ultrametric space an open ball is automatically closed you are essentially done.

As a totally disconnected space, $\mathbb{Z}_p$ cannot be locally Euclidian and hence cannot be isomorphic to a Lie group. However, $\mathbb{Z}_p$ is "almost" a Lie group in the sense that it becomes a Lie group after a minor modification, namely after dividing out the compact subgroups $p^j\mathbb{Z}_p$. One obtains the cyclic groups $\mathbb{Z}/p^j\mathbb{Z}$ which are discrete and therefore Lie groups. In general, the *Gleason-Yamabe theorem*[2] specifies how locally compact topological groups have to be modified in order to become Lie groups.

[2] See e.g. *Hilbert's Fifth Problem and Related Topics* by Terry Tao.