# Lie groups and Representation Theory - Lecture 7

## Representation theory of compact topological groups II - The Peter-Weyl theorem

In the previous lecture, we established existence and uniqueness of (left and right) Haar measure $\mu$ on any compact Hausdorff topological group $G$. For us today, the upshot of the construction is that we have a class of functions $f: G \to \mathbb{C}$ (the measurable functions) that we can integrate, and that we have

$$\int f(x)\, d\mu(x) = \int L_y f\,(x)\, d\mu(x) = \int R_y f\,(x)\, d\mu(x)$$

$$= \int f(x^{-1})\, d\mu(x)$$

where
$$L_y f\,(x) = f(y^{-1}x)$$
$$R_y f\,(x) = f(xy)$$
are the left- and right regular representations of $G$.

Our aim today is to show that the abstract representation theory of $G$ is essentially similar to that of a finite group: unitarity, complete reducibility, decomposition of the regular representation.

For this enterprise, it turns out wise (or sufficient, depending on standpoint) to restrict the left-regular representation to the Hilbert space of square-integrable functions on $G$.

$$L^2(G,\mu) = \left\{ f : G \to \mathbb{C}, \quad \int_G |f(x)|^2 \, d\mu(x) < \infty \right\}$$

with inner product

$$\langle f, g \rangle = \int_G \bar{f}(x) \, g(x) \, d\mu(x) \in \mathbb{C} \quad \text{for } f, g \in L^2(G,\mu).$$

The point is that because of invariance of Haar measure, the left-regular representation is unitary w.r.t. this inner product:

$$\langle L_y f, L_y g \rangle = \int_G \bar{f}(y^{-1}x) \, g(y^{-1}x) \, d\mu(x)$$

$$= \int_G \bar{f}(x) \, g(x) \, d\mu(x)$$

$$= \langle f, g \rangle.$$

(in particular $L_y : L^2(G,\mu) \circlearrowleft$ ).

The main result is the celebrated Peter-Weyl theorem, which says that as (unitary) representation of $G$,

$$L^2(G,\mu) = \bigoplus_a V_a \otimes V_a^*$$

where the (in general, infinite!) sum is over all (isomorphism classes of) finite-dimensional irreducible (unitary) representations $(V_a, \rho_a)$ of $G$.
So really the only difference to the finite group case is that we have countably many iso-classes of irreps.
We'll formulate the theorem in a slightly different way, so as to be able to deduce as another application that all unitary representations are completely reducible to finite-dimensional pieces. (and in particular unitary irreps are finite-dimensional).

⌈ I have been trying to ascertain whether there are any non-unitary infinite dimensional irreps of compact groups. The fact that ~~[illegible]~~ $L^2(G) \subsetneq L^1(G)$ seems to suggest that there is room for such things. ⌋

For finite-dimensional representations, unitarity is automatic:

⌈ I am now following the strategy of B. Simon ⌋

<u>Lemma:</u> Let $\rho: G \to GL(V)$ be a finite-dimensional representation of the compact topological group $G$ with Haar measure $\mu$. Assume that for all $v \in V$, $\lambda \in V^*$, the representation function

$$G \ni x \longmapsto \lambda(\rho(x)v) \in \mathbb{C}$$

is measurable and bounded and that for all $v \in V$, the function $(v \neq 0)$

$$G \ni x \mapsto \rho(x)v \in V$$

is not "zero for almost every (a.e.) $g$".

Then there exists an inner product on $V$ w.r.t. which $\rho$ is unitary.

<u>Pf:</u> Is the same as in finite group case, modulo the technical analytic assumptions. Let $(\cdot,\cdot)$ be any inner product on $V$, and define for $v,w \in V$,

$$\langle v,w \rangle = \int_G (\rho(x)v, \rho(x)w)\, d\mu(x)$$

this is well-defined because e.g.

$$(\rho(x)v, \rho(x)w) = \sum_{i=1}^{\dim V} (\rho(x)v, e_i)(e_i, \rho(x)w)$$

where $e_i \in V$ are orthonormal wrt $(\cdot,\cdot)$

and $(e_i, \rho(x)w)$ is measurable and bounded.

If $v \neq 0$, $\rho(x)v$ is not zero for a.e. $x$, so $(\rho(x)v, \rho(x)v) \geq 0$ which is non-negative is not zero for a.e. $x$. So

$$\langle v, v \rangle > 0 \quad \text{if } v \neq 0$$

$\square$

Another technicality that we should in principle worry about is continuity (since $G$ is topolgical group).

Lemma: Let $\rho : G \to GL(V)$ be **finite-dimensional** unitary representation of compact topological gp $G$ s.t. representation functions

$$x \longmapsto \lambda(\rho(x)v) \in \mathbb{C}$$

are measurable $\forall v \in V$, $\lambda \in V^*$. Then the representation is continuous.

Pf: Uses the probably familiar to you "smoothing by convolution" trick.
Note **that** since $V$ is finite-dimensional, any two norms on $GL(V)$ will be equivalent.
We prove that the representation functions are continuous.
For $w \in V$, $\lambda \in V^*$, and $f \in C(G)$, consider the function on $G$,

$$s(x) = \int f(y^{-1}x) \, \lambda(\rho(y)w) \, d\mu(y)$$

Since $f$ is uniformly continuous and $\rho(y)$ is unitary hence $\lambda(\rho(y)W)$ is bounded, so $s$ is continuous in $x$

[ If $x^{-1}z \in V$ neighborhood of identity s.t. $|f(x)-f(z)|<\varepsilon$ for $x^{-1}z \in V$ then

$$|s(x)-s(z)| \leq \int |f(y^{-1}x)-f(y^{-1}z)|\,|\lambda(\rho(y)W)|\,d\mu(y)$$

$$< \varepsilon \qquad \text{since } (y^{-1}x)^{-1}y^{-1}z = x^{-1}z.$$ ]

On the other hand,

$$s(x) = \int f(y^{-1}x)\,\lambda(\rho(y)W)\,d\mu(y)$$

$$= \int f(y^{-1})\,\lambda(\rho(xy)w)\,d\mu(y)$$

$$= \lambda\left(\rho(x)\int f(y^{-1})\,\rho(y)\,w\,d\mu(y)\right)$$

So $\lambda(\rho(x)v)$ is continuous in $x$ for all $v$ of the form

$$v = \int f(y^{-1})\rho(y)\,w\,d\mu(y) \qquad w \in V,\ f \in C(G).$$

If $\exists\, v \neq 0$ not of that form, there exists $\lambda \neq 0$ s.t.

$$\lambda\left(\int f(y^{-1})\rho(y)\,w\,d\mu(y)\right) = 0 \qquad \forall\, w, f$$

$$= \int f(y^{-1})\ \lambda(\rho(y) w)\ d\mu(y)$$

Since $C(G)$ is dense in $L^2(G)$ we find

$$\lambda(\rho(y) w) = 0 \qquad \text{for a.e. } y, \text{ for all } w \in V.$$

So there is $y$ s.t. $\lambda(\rho(y) w) = 0$ $\forall w \in V$. Since $\rho(y)$ is unitary, this implies $\lambda = 0$.

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- These lemmas show that we don't miss much if we don't assume unitarity and continuity and also highlight the role of the representation functions.
- The Peter–Weyl theorem shows that in a sense we don't miss anything.

- As a side remark, ~~of~~ course we define irreducibility and isomorphism of reps as in the finite group case, and complete reducibility and Schur's lemma hold as in finite group case.

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~~Peter–Weyl theorem: Let $G$ be a compact topological group with Haar measure $\mu$. Let $(V_a)_{a \in A}$ be the isomorphism classes of irreducible unitary continuous repres~~

**Peter-Weyl theorem:** Let $G$ be a compact Hausdorff topological group with Haar measure $\mu$. Let $(V_a, \rho_a)_{a \in A}$ be the collection of isomorphism classes of irreducible continuous unitary finite-dimensional representations of $G$. Then the collection of representation functions

$$y \mapsto \lambda(\rho_a(y)v) \qquad v \in V_a,\ \lambda \in V_a^*$$

are dense in $C(G)$ (with respect to sup-norm). If for each $a$,

$$\left(e_i^{(a)}\right)_{i=1}^{n_a} \qquad n_a = \dim V_a$$

is orthonormal basis of $V_a$ w.r.t. $\langle \cdot, \cdot \rangle$, then the functions

$$y \mapsto \sqrt{n_a}\, \langle e_i^{(a)}, \rho_a(y)\, e_j^{(a)} \rangle$$

are orthonormal (Hilbert-space) basis of $L^2(G,\mu)$.

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We'll prove both statements at the same time, and also the related

$$L^2(G,\mu) = \bigoplus_a V_a \otimes V_a^*$$

The proof depends on a piece of functional analysis (about Hilbert-Schmidt operators) that I'll quote when we get to it.

We begin by observing the following consequence of translational invariance of Haar measure:

Let $V_1, V_2$ be two (..) representations of $G$, and $B \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$.

Then
$$\bar{B} := \int_G \rho_2(y) B \rho_1(y^{-1})\, d\mu(y)$$
satisfies
$$\bar{B}\rho_1(x) = \rho_2(x)\bar{B}$$
i.e. $\bar{B} \in \mathrm{Hom}_G(V_1, V_2)$.

In particular, if $V_1 = V_a$, $V_2 = V_b$ $(a, b \in A)$ are irreducible, then by Schur's lemma
$$\bar{B} = \alpha\, \delta_{ab}\, id_{V_a}$$
and we can calculate $\alpha$ by taking the trace to be
$$\alpha = \frac{\operatorname{tr} B}{\dim V_a}$$

For instance, if $(e_i^{(a)}) \subset V_a$, $(e_j^{(b)}) \subset V_b$ are an orthonormal basis, we may define for each $i, j$,

$$B : V_a \to V_b$$

$$B(v) = e_j^{(b)} \langle e_i^{(a)}, v \rangle \qquad [\operatorname{tr} B = 1 \text{ if } a = b]$$

and we learn

$$\int \rho_b(y^{-1})\, e_j^{(b)} \langle e_i^{(a)}, \rho_a(y) v \rangle = \frac{\delta_{ab}\,\delta_{ij}}{\dim V_a}\, v$$

taking $v = e_k^{(a)}$ and calculating inner product with $e_\ell^{(b)}$, we find

$$\int \langle e_\ell^{(b)}, \rho_b(y^{-1})\, e_j^{(b)} \rangle \langle e_i^{(a)}, \rho_a(y)\, e_k^{(a)} \rangle \, d\mu(y)$$

$$= \frac{\delta_{ab}\,\delta_{ij}\,\delta_{k\ell}}{\dim V_a}$$

Since $\langle e_\ell^{(b)}, \rho_b(y^{-1})\, e_j^{(b)} \rangle = \overline{\langle e_j^{(b)}, \rho_b(y)\, e_\ell^{(b)} \rangle}$

this is tantamount to the orthonormality of the representation functions above.

- Remains to prove completeness.

→ Also remark that orthogonality of characters follows at once from this.

- The fact that $L^2(G,\mu) = \bigoplus V_a^{\oplus \dim V_a}$ follows from this orthogonality and completeness, as can be seen by formal matrix manipulations.

The fact that $L^2(G,\mu) = \bigoplus V_a^{\dim V_a}$ follows basically from orthogonality and completeness, which we establish below.

The fact that any irrep $(V,\rho)$ occurs $\dim V$ times in $L^2(G,\mu)$ may also be seen more formally as follows:

For any $\lambda \in V^*$ define the linear map

$$\mathcal{U} : V \to L^2(G,\mu)$$

$$\mathcal{U}(v)\,(y) = \sqrt{\dim V}\;\; \lambda(\rho(y^{-1})v)$$

One can check easily that this intertwines $\rho$ and the left regula representation : $\mathcal{U}(\rho(x)v)(y) =$

$$\mathcal{U}(\rho(x)v)(y) = \sqrt{\dim V}\;\lambda(\rho(y^{-1})\rho(x)v) = \sqrt{\dim V}\;\lambda(\rho(y^{-1}x)v)$$

$$= \sqrt{\dim V}\;\lambda(\rho((x^{-1}y)^{-1})v) = \mathcal{U}(v)(x^{-1}y).$$

$$= \cancel{\sqrt{\dim V}}\;(L_x\,\mathcal{U}(v))\,(y).$$

Unitarity can also be seen by the above argument:

$$\int \overline{\mathcal{U}(v)(y)}\;\;\mathcal{U}(w)(y)\;d\mu(y)$$

$$= \dim V \int \overline{\langle \ell, \rho(y^{-1})v\rangle}\,\langle \ell, \rho(y)w\rangle\; d\mu(y)$$

$$= \dim V \int \langle v, \rho(y) \underbrace{l \rangle \langle l}_{B: V \to V,\ \mathrm{tr} B = 1}, \rho(y) w \rangle \, d\mu(y)$$

$$= \langle v, w \rangle$$

where $l \in V$ is such that $\langle l, v \rangle = \lambda(v) \ \forall v \in V$.

- Injectiveness of $\mathcal{U}$ follows also more directly from Schur's lemma by noting that if $\lambda(\rho(y^{-1})v) = 0 \ \forall y$, then either $v = 0$ or $\mathrm{Ker}(\lambda)$ is an invariant subspace of $V$.

- Similarly, independence of $\mathcal{U}(v)$ for different $\lambda$ follows also from orthogonality.

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~~Conversely~~ ~~let~~ To show that $\mathbb{V}$ does not occur more often than $\dim V$ times, one could proceed as follows (~~th~~ in the finite group case, we could use characters to calculate multiplicities. Here however the left-regular representation is infinite dimensional, and the notion of character for it is not obvious), though morally,

$$\int d\mu(x) \ \mathrm{tr}_{L^2(G,\mu)} L_x = 1$$

replaces $\frac{1}{|G|} \sum_g \mathrm{tr}_{\mathbb{C}G} L_g = \frac{1}{|G|} \mathrm{tr}_{\mathbb{C}G} L_e = \frac{|G|}{|G|} = 1$

- Let $\mathcal{U} : V \to L^2(G, \mu)$ be a "unitary intertwiner" between irrep $(V, \rho)$ and the left-regular representation, which means a partial isometry, a linear in $v$ assignement

$$v \mapsto \mathcal{U}_v \in L^2(G, \mu)$$

satisfying

$$\mathcal{U}_v(x^{-1}y) = (L_x \mathcal{U}_v)(y) = \mathcal{U}_{\rho(x)v}(y).$$

- To show that $\mathcal{U}_v(y) = \lambda(\rho(x)v)$ for some $\lambda \in V^*$, we evaluate at the identity, yielding a $\lambda$ s.t.

$$\mathcal{U}_v(e) = \lambda(v) \quad \forall v \in V.$$

- In order for evaluation at identity to make sense, we have to ensure that the $\mathcal{U}_v$ are all continuous functions on $G$.
This can be done in the same spirit as before by "smoothing by convolution", and an approximation of identity. - If $f \in C(G)$, then

$$\int f(yx)\, \mathcal{U}_v(x^{-1})\, d\mu(x)$$

is continuous in $y$. On the other hand, it equals

$$\int f(x)\, \mathcal{U}_v(x^{-1}y)\, d\mu(x)$$

$$= \int f(x)\, \mathcal{U}_{\rho(x)v}(y)\, d\mu(x).$$

which means that this function is in the image of $U$.

By "approximation of identity in $C(G)$" (i.e. a sequence of functions convolution with which converges to identity operator, basically a continuous approximation of Dirac $\delta$), one concludes that continuous functions are dense in the image of $U$, which since $\mathrm{Im}(U)$ is finite-dimension implies that all $U_v$ are continuous.

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The proofs that representation functions are dense (in $C(G)$ and $L^2(G,\mu)$) also use such approximation of identity as well as the properties of Hilbert-Schmidt integral operators.
I'll reproduce here the proof in B. Simon's book, as it seems the most elementarily accessible.

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~~The basic idea is the following:~~
The starting point is the following: For every continuous function $g \in C(G), \subset L^2(G,\mu)$, consider the integral operator

$$T_g : L^2(G,\mu) \to L^2(G,\mu)$$

$$T_g f(y) = g * f(y) = \int g(x^{-1}y)\, f(x)\, d\mu(x)$$

- Because $G$ is compact, this operator is <u>Hilbert-Schmidt</u> i.e.

$$\underbrace{\mathrm{tr}_{L^2(G,\mu)}\, T_g^* T_g}_{} < \infty$$

$$\left( = \int\int |g(x)|^2 \, d\mu(x)\, d\mu(y) < \infty \right)$$

- If $g(x) = \overline{g(x^{-1})}$, $T_g$ is self-adjoint

- By the spectral theorem for self-adjoint Hilbert-Schmidt operators (which are in particular compact), there is a sequence of eigenvalues $0 \neq \lambda_n \to 0$, and finite-dimensional eigenspaces $H_n$ $\left( T_g|_{H_n} = \lambda_n \cdot \mathrm{id}_{H_n} \right)$ such that

$$(\mathrm{Ker}\, T_g)^\perp = \bigoplus_{n=1}^{\infty} H_{\lambda_n}$$

- Since as we've used before $g$ being uniformly continuous and

$$H_{\lambda_n} = \frac{1}{\lambda_n} T_g H_{\lambda_n} \qquad (\lambda_n \neq 0)$$

the non-zero eigenspaces consist of continuous functions

- Since $T_g L_x = L_x T_g$ $\forall x \in G$, the $H_{\lambda_n}$ are invariant subspaces of left-regular representation.

- By complete reducibility, each $H_{\lambda_n}$ is isomorphic to direct sum of irreps, hence is spanned by our representation functions.

- Therefore, any function $T_g f \in C(G) \subset L^2(G,\mu)$ in image of $T_g$ is in the Hilbert space span of the representation functions, i.e.

$$T_g f = \lim_{k\to\infty} {}_{\|\cdot\|_2} f_k \qquad f_k \in \bigoplus H_{\lambda_n}$$

- By applying $T_g$ again, and using that

$$\| T_g f \|_\infty \leq \|g\|_\infty \, \|f\|_2$$

$$\left( \left| \int g(x^{-1}y) f(x)\, d\mu(x) \right| \leq \sup g \cdot \int |f| \underset{\text{by C-S}}{\leq} \|g\|_\infty \cdot \|f\|_2 \right)$$

we conclude that

$$T_g^2 f = \lim_{k\to\infty} {}_{\|\cdot\|_\infty} T_g f_k \qquad T_g f_k \in \bigoplus H_{\lambda_n}.$$

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So finally, if $f \in C(G)$ is continuous, then $f$ is uniformly continuous, i.e. $\forall \varepsilon \; \exists \; A$ nbhd of $e$ s.t.

$$| f(x) - f(y) | < \varepsilon$$

Now let's conclude. Given $f \in C(G)$, we wish to approximate it uniformly by representation functions.

Let $\varepsilon > 0$. Since $f$ is uniformly continuous, $\exists A$ neighborhood of $e$ s.t. $|f(x) - f(z)| < \varepsilon$ $x^{-1}z \in A$.
Let $B \ni e$ be nbhd s.t. $B^2 \subset A$ $B = B^{-1}$, and let $g$ be positive continuous function with support in $B$, $g(x) = g(x^{-1})$ and $\int g(x)\, d\mu(x) = 1$. (Such a function exists)

Then

$$| T_g^2 f(y) - f(y) |$$

$$= \left| \int d\mu(x) d\mu(z) \quad g(x^{-1}y)\, g(z^{-1}x) \big(f(z) - f(y)\big) \right|$$

$$\leqslant \int d\mu(x)\, d\mu(z) \quad g(x^{-1}y)\, g(z^{-1}x) \quad | f(z) - f(y) |$$

Now integrand is only non-zero if $x^{-1}y$, $z^{-1}x \in B$ where $z^{-1}x\ x^{-1}y = z^{-1}y \in A$, hence $|f(z) - f(y)| < \varepsilon$

So $\quad \| T_g^2 f - f \|_\infty < \varepsilon$

By the previous considerations, $\exists$ representation function $r$ s.t. $\| T_g^2 f - r \|_\infty < \epsilon$, hence by triangle inequality

$\| r - f \|_\infty < \varepsilon$ and we are done.