Lie groups and representation theory - Lecture 6

Compact topological groups I - Haar measure

Def: A topological group is a group G endowed with a topology (i.e. a system of subsets called "open" that includes open set and all of G and is closed under arbitrary union and finite intersection) such that group operations

$$G \times G \to G \qquad (x,y) \mapsto x \cdot y$$
$$G \to G \qquad x \mapsto x^{-1}$$

(equivalently $(x,y) \mapsto xy^{-1}$) are continuous. (i.e. preimages of open sets are open}, note that here this is the same as being open maps.)

Def: A compact topological group is a topological group that is a compact topological space (defined by Heine-Borel property, that every open cover has a finite subcover.)

Rem: We'll always assume (implicitly) that our topology is Hausdorff. (though this can always be repaired as we show presently). Our goal is to turn such a compact topological group into a measure space in a canonical fashion.

Ad compactness

For metric spaces:

Bolzano-Weierstrass (every sequence has convergent subsequence)

$\Updownarrow$

Complete and totally bounded (every Cauchy sequence converges and cover by finitely many balls of radius ε)

$\Updownarrow$

Heine-Borel (every open cover has finite subcover)

- In general topological spaces, does not play same role, * so it's best to define compactness using Heine-Borel. (* Consider $\mathbb{C}^{\mathbb{R}}$, complex functions on $\mathbb{R}$, with product topology. Sequences converge if they converge pointwise. but this does not come from a metric. Closure of ~~B~~ $C(\mathbb{R})$ in $\mathbb{C}^{\mathbb{R}}$ is all of $\mathbb{C}^{\mathbb{R}}$ (every open neighborhood contains continuous function) but pointwise convergent ~~of~~ sequence of continuous functions is Borel measurable.)

- ~~However~~ ~~if~~ We ~~will~~ ~~mostly~~ use Heine-Borel. ~~B~~ to define compact topological groups. In construction of Haar measure, we consider $C(G)$ with sup norm, which is a complete metric space, so these other ~~definitions~~ characterization work as well.

Let's begin with two results that I am actually able to prove in an elementary fashion and at least the second of which will be useful in the construction of Haar measure. The first is to give a flavor of the interaction between topology & gp. structure.

Proposition: Let G be a topological group.

(i) If points are closed in G (i.e. G is T_1) then G is Hausdorff (i.e. T_2).

(ii) If G is not T_1, let H be the closure of $\{e\}$. Then H is a normal subgroup of G and G/H equipped with the quotient topology is T_1, hence a Hausdorff topological group.

Proof: (i) The basic idea is to exploit translational invariance (i.e. U is open $\Leftrightarrow$ xU is open for all $x \in G$, this follows from continuity).

- We have to show that for any $x \neq y$, $\exists\ U \ni x$, $V \ni y$ open with $U \cap V = \emptyset$.
- By translational invariance, it is enough to show this for $y = e$.
- Since G is T_1, $V = G \setminus \{x\}$ is open (nbdd of e)
- By continuity of multiplication,

$$m : G \times G \to G$$

$m^{-1}(V)$ is an open neighborhood of $(e,e) \in G \times G$

Lemma: Let G be a topological group. For any $x \in G$ the map

$$m(x,\cdot) : G \to G$$

$$y \mapsto m(x,y) = x\cdot y$$

is continuous

Proof: Let W be an open neighborhood of xy in G. By continuity of $m: G\times G \to G$, the preimage $m^{-1}(W)$ is open in $G\times G$. By the definition of product topology, there exist open neighborhoods U of x and V of y such that

$$U\times V \subset m^{-1}(W).$$

But then $x\cdot V \subset W$, so $V \subset m(x,\cdot)^{-1}(W)$, which implies that $m(x,\cdot)$ is continuous.

- $G \times G$ being equipped with product topology means that $m^{-1}(V)$ being open means that it contains an open neighborhood of the form $\mathcal{U}_1 \times \mathcal{U}_2$ with $\mathcal{U}_1, \mathcal{U}_2$ open neighborhoods of e in G.
- Now $\tilde{\mathcal{U}} = \mathcal{U}_1 \cap \mathcal{U}_2$ satisfies $\tilde{\mathcal{U}} \cdot \tilde{\mathcal{U}} \subset V$, and $\mathcal{U} = \tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}^{-1}$ satisfies in addition $\mathcal{U} = \mathcal{U}^{-1}$, $\mathcal{U} \cdot \mathcal{U} \subset V \not\ni x$.
  So $\mathcal{U}$ is an open neighborhood of e, and $x\mathcal{U}$ an open neighborhood of x.
  $$\mathcal{U} \cap x\mathcal{U} = \emptyset$$
  since otherwise $x u_1 = u_2 \Rightarrow x = u_2 u_1^{-1} \in \mathcal{U}\mathcal{U}^{-1} \subset V$

(ii) By definition H is the intersection of all closed subsets containing e, or in other words

$$G \setminus H = \bigcup_{\mathcal{U} \not\ni e} \mathcal{U}$$

We claim that H is a subgroup of G. Indeed, if $x \in H$, $y \in H$, but $xy \notin H$, $\exists$ open neighborhood $y \in W \not\ni e$. But $m^{-1}(W)$ is open neighborhood of $(x, y) \in G \times G$ so contains $\mathcal{U} \times V$ with $x \in \mathcal{U}$, $y \in V$. But e must be in both $\mathcal{U}$ and V (otherwise ~~they~~ x, y would not be in H) so $e \in m(\mathcal{U} \times V) = W$, which is a contradiction. Similarly for the inverse.

- H is normal because for any $x \in G$, $xHx^{-1} \ni e$ is a closed subset, so $H \subset xHx^{-1}$.
This is also true for x^{-1}, so

$$H \subset xHx^{-1} \subset xx^{-1}Hxx^{-1} = H$$

so $xHx^{-1} = H$.

- To show that $\bar{e} = eH = H$ is closed in G/H, we note that $G/H \setminus \{\bar{e}\}$ is open because its inverse image under the projection $G \to G/H$ is $G \setminus H$, which is open.

- Verification of continuity of multiplication and inverse follows similarly.

$\square$

[The prop. also gives a flavor of the arguments involved in Hilbert 5^{th}]

The second simple result which we'll need for the Haar measure is the following

~~Proposition: Let $f: G \to \mathbb{C}$ be a continuous function.~~

on a ~~topologi~~

<u>Proposition:</u> Let G be a compact topological group and $f : G \to \mathbb{C}$ be a continuous function ($f \in C(G)$). Then f is uniformly continuous, in the sense that for every $\varepsilon > 0$, there exists a (symmetric) neighborhood $V \ni e$ ($V^{-1} = V$) such that

$$|f(x) - f(y)| < \varepsilon$$

$\forall \; x, y \in G$ s.t. $x^{-1}y \in V$ or $yx^{-1} \in V$.

<u>Proof:</u> Note first of all that the statement is equivalent to

$$|f(xz) - f(x)| < \varepsilon$$

$$|f(zx) - f(x)| < \varepsilon$$

$\forall \; x \in G, \quad z \in V$.

To establish this, we consider the function

$$F : G \times G \longrightarrow \mathbb{C}$$

$$F : (x, z) \longmapsto f(xz) - f(x).$$

This function is continuous at $(x, e) \in G \times G$ for every $x \in G$. This means that $\forall x \in G$ there exists $U_x \ni x$, $V_x \ni e$

such that

$$|F(x',z)| < \varepsilon$$

$\forall \quad (x',z) \in \quad U_x \times V_x \quad , \quad$ wolg $\quad V_x = V_x^{-1}$.

Since $\quad \bigcup_{x \in G} U_x = G \quad$ and G is compact,

there are finitely many $(x_i)_{i=1,..N}$ s.t.

$G = \bigcup_i U_{x_i}$.

Let $\quad V = \bigcap_i V_{x_i} \ni e$.

Then, if $x' \in G$, and $z \in V$, $\quad x' \in U_{x_i}$ for some i

and $z \in V_{x_i}$, so

$$|F(x',z)| = |f(xz) - f(x)| < \varepsilon \qquad \square.$$

[The ~~left~~ translation property can be ensured afterwards, or directly by considering

$$|f(xz) - f(x)| \quad + \quad |f(zx) - f(x)|$$

]

- Our goal now is to take such a compact topological group G and turn it into a measure space. (I'll first say what, then why, then how.)

- Now most of you, though presumably not everyone, have seen measure spaces before. For those who haven't, the idea is that we want to be able to measure sizes $\mu(E) \in [0,\infty]$ of suitable subsets $E \in \mathcal{B} \subset \mathcal{P}(G) = (\mathbb{Z}_2)^G$ (technically $\mathcal{B}$ a σ-algebra, closed under countable union and complement) such that

$$\mu(\phi) = 0$$

$$\mu(\underbrace{\cup E_i}_{\text{countable disjoint union.}}) = \sum \mu(E_i)$$

- G already having a topology means that there is a prejudice for measures in which open sets are measurable (these are so-called Borel measures).

- Most importantly, G having a group structure allows us to distinguish measures which are invariant under translations. In principle, we can distinguish

left-translational invariance $\mu(xE) = \mu(E)$

right-translational invariance $\mu(Ex) = \mu(E)$

• Perhaps least importantly for us, but technically certainly very relevant, is a certain regularity assumption w.r.t. exhaustion:

- $\mu(E) = \inf\{\mu(U),\ E \subset U \text{ open}\}$
- $\mu(E) = \sup\{\mu(K),\ E \supset K \text{ compact}\}$
- $\mu(K)$ finite when K is compact

This is the regularity for G compact.

In particular, it makes sense to "normalize" μ by ~~setting~~ condition $\mu(G) = 1$.

Def: Let G be a compact topological group. A left/right Haar measure is a regular Borel measure on G that is left/right translationally invariant.

Thm: A left Haar measure exists and is unique up to overall scale.
(which we will normalize to $\mu(G) = 1$, remark that this is not necessarily natural from differential geometric point of view).

Proof: Later.

<u>Remark/Corollary</u> If μ is a/the left Haar measure, then $\tilde{\mu}$, defined by

$$\tilde{\mu}(E) = \mu(E^{-1})$$

satisfies $\tilde{\mu}(Ex) = \mu(x^{-1}E^{-1}) = \mu(E^{-1}) = \tilde{\mu}(E)$

so is a right Haar measure, which then is also unique in particular.

Moreover, if $y \in G$, then

$$\mu_y(E) = \mu(Ey)$$

is also a left Haar measure, so by uniqueness $\exists$ function $\Delta : G \to \mathbb{R}^+$ s.t.

$$\mu_y = \Delta(y)\,\mu$$

Since clearly $\mu(Gy) = \mu(G)$, $\mu(Gy) = \Delta(y)\mu(G)$ and $\mu(G) = 1$ implies $\Delta(y) = 1$ $\forall y$ so in fact

$$\mu(Ey) = \mu(E)$$

i.e. μ is also a/the right Haar measure and satisfies $\mu(E^{-1}) = \mu(E)$.

<u>Remark/Warning:</u> For us, G is always compact and Hausdorff.
A left Haar measure also exists and is unique when G is merely locally compact Hausdorff.
A right Haar measure also exists but left and right need not agree.

Why? • In general, one of the motivations for constructing measures on spaces is not just to measure sizes of subsets, but also to integrate functions, which is to assign a (complex) number

$$\mu(f) = \int_G f(x)\, d\mu(x) = \int f$$

to any suitable (namely measurable) function $f: G \to \mathbb{C}$.

• The main characteristic properties of the integral of course being

$(\lambda \in \mathbb{C})$

• $$\int \lambda f + g = \lambda \int f + \int g \quad \text{(linearity)}$$

• $$\left| \int f \right| \leq \int |f| \quad \text{(estimate)}$$

or alternatively, $\int f \geq 0$ if $f \geq 0$ (monotonicity)

• and importantly for us

$$\int L_x f = \int f = \int R_x f$$

(translational invariance)

$$L_x f(y) = f(x^{-1}y).$$

• This will remind you of why we need the Haar measure. It is to pull the tricks that were so helpful in representation theory of finite groups,

- unitarity,
- projection formulas

→ The remark that makes the link is that counting measure is a Haar measure on a discrete group.

$$\sum_g f(g) = \sum_g f(hg)$$

In general,

- To go from the measure to the integral, one proceeds by exhaustion by characteristic functions of measurable sets.

- More precisely, to integrate complex functions, one separates real and imaginary parts, and for a real function, positive and negative parts, while for a positive function

$$\mu(f) = \sup \left\{ \int \sum_{i=1}^{N} a_i \chi_{E_i} ,\ 0 \leq \sum a_i \chi_{E_i} \leq f \right\}$$

- Conversely, in the LCH case, one can go back from positive linear functionals on $C(G)$ to measures on G. This is the content of the Riesz representation theorem, which we'll treat as "black box."

__Riesz representation theorem__ Let X be compact Hausdorff space, and $I: C(X) \to \mathbb{C}$ be a positive linear functional on the space of continuous functions, i.e.

- $I(\lambda f + g) = \lambda I(f) + I(g) \quad f, g \in C(X) \quad \lambda \in \mathbb{C}$
- $I(f) \geq 0 \quad$ if $\quad f \geq 0, \quad f \in C(X)$.

Then there exist a unique regular Borel measure μ on $\mathcal{B}X$ such that

$$I(f) = \int_X f(x)\, d\mu(x) \qquad \forall\; f \in C(X).$$

__Remark:__ To be sure, when $X = G$ is a Lie group, there are other ways of constructing Haar measure, in particular to pull it back from Lebesgue measure on $\mathbb{R}^n$. Nevertheless, it's instructive to give

Proof of existence and uniqueness of left Haar measure for compact topological groups via Riesz representation theorem, and another probably familiar result, the

__Arzela-Ascoli Theorem__ ⊛ If $\mathcal{F}$ is an equicontinuous pointwise bounded subset of $C(X)$, then $\mathcal{F}$ is totally bounded in uniform metric on $C(X)$ and the closure of $\mathcal{F}$ is compact. In particular, every sequence in $\overline{\mathcal{F}}$ contains a uniformly convergent subsequence. (Bolzano - Weierstrass).

⊛ X a compact Hausdorff space. Equicontinuous will be defined below - perhaps put it here...

<u>Equicontinuous</u> For every $x \in X$, and every $\varepsilon > 0$, $\exists$ $\mathcal{U}$ open neighborhood of x, s.t.

$$|f(x) - f(y)| < \varepsilon$$

$\forall\, y \in V$, and all $f \in \mathcal{F}$.

The important point is that $\mathcal{U}$ does not depend on f. ~~is compact~~

Note that if $X = G$ is compact Hausdorff topological group, equicontinuity automatically improves to uniform equicontinuity.

<u>Totally bounded</u>. For every $\varepsilon > 0$ $\mathcal{F}$ can be covered by finitely many balls of radius ε.

For metric spaces:

Bolzano-Weierstrass $\Leftrightarrow$ Heine-Borel $\Longleftrightarrow$ complete and totally bounded.

- As just became clear, the key is to exploit certain properties of the space $C(G)$ of continuous functions which when equipped with the uniform ~~metr~~ norm as metric is a complete metric space.

- Since any positive linear functional is continuous on $C(G)$ (w.r.t. uniform metric – this follows for example from the estimate $|\int f| = \alpha \int f = \mathrm{Re} \int \alpha f = \int \mathrm{Re}(\alpha f) \leq \int |\mathrm{Re}(\alpha f)| \leq \int |\alpha f| = \int |f| \leq \sup |f| \cdot \int 1$, where $|\alpha| = 1$), it's enough to show that there exists a positive linear functional $I: C(G) \to \mathbb{C}$ satisfying

$$I(L_x f) = I(f) \qquad I(1) = 1$$

and that if $\tilde{I}$ is a continuous linear functional satisfying

$$\tilde{I}(L_x f) = \tilde{I}(f) \qquad \tilde{I}(1) = 1$$

then $\tilde{I} = I$. (If $\tilde{\mu}$ is a left-invariant measure, define $\tilde{I} = \tilde{\mu}|_{C(G)}$ then $\tilde{I} = I$ implies $\tilde{\mu} = \mu$)

- Because of complex linearity, it's enough to construct $I(f)$ for real valued functions.

So consider the complete metric space $C(G,\mathbb{R})$ of continuous real-valued functions on the compact Hausdorff topological group G. On $C(G,\mathbb{R})$, we have the continuous functions

$$M(f) = \max(f)$$

$$m(f) = \min(f)$$

$$v(f) = M(f) - m(f).$$

Now if $f \in C(G,\mathbb{R})$, consider the subset of $C(G,\mathbb{R})$,

$$\mathcal{L}(f) = \left\{ \sum_{i=1}^{n} a_i L_{x_i} f,\ n \in \mathbb{N},\ a_i > 0,\ \sum a_i = 1,\ x_i \in G \right\}$$

where $L_{x_i} f(x) = f(x_i^{-1} x)$.

- For any $y \in G$, $\mathcal{L}(L_y f) = \mathcal{L}(f)$
- If $g = \sum_{i=1}^{n} a_i L_{x_i} f \in \mathcal{L}(f)$, then

(i) $\mathcal{L}(g) \subseteq \mathcal{L}(f)$

(ii) $g \geq 0$ if $f \geq 0$.

(iii) $\|g\|_\infty \leq \sum a_i \|f\|_\infty = \|f\|_\infty$

Since f is uniformly continuous, for every $\varepsilon>0$ $\exists\, V \ni e$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \text{if } x^{-1}y \text{ or } yx^{-1} \in V.$$

Then for any $g \in \mathcal{L}(f)$ we have

$$\text{(iv)} \quad |g(x) - g(y)| = \left| \sum a_i \left(L_{x_i} f(x) - L_{x_i} f(y) \right) \right|$$

$$\leq \sum a_i \, |L_{x_i} f(x) - L_{x_i} f(y)|$$

$$< \varepsilon$$

if $x^{-1}y \in V$ (since then $(x_i^{-1}x)^{-1} x_i^{-1} y = x^{-1}y \in V \; \forall i$)

We have just proven (iii) & (iv) that $\mathcal{L}(f)$ is bounded and equicontinuous. By Arzela-Ascoli, this implies that $\overline{\mathcal{L}(f)}$ (closure of $\mathcal{L}(f)$) is a compact subset of $C(G, \mathbb{R})$. Since v = max - min is continuous, it assumes a minimum on $\overline{\mathcal{L}(f)}$, say at ~~$C(G,\mathbb{R})$~~. $f_* \in \overline{\mathcal{L}(f)}$.

The claim is that $v(f_*) = 0$, i.e. f_* is a constant function, and our plan is to define

$$I(f) = f_*(e)$$

as the "integral of f".

Pf of claim: ($\overline{\mathcal{L}(f)}$ contains a constant function)

- If on the contrary, $v(f_*) \neq 0$, i.e. $M(f_*) > m(f_*)$ then by continuity, there exits a non-empty open set U on which

$$f_* > m(f_*) + \frac{v(f_*)}{2} = \frac{M(f_*) + m(f_*)}{2}$$

- Since G is compact, finitely many translates $\{x_i U\}_{i=1}^n$ cover G. Consider

$$\tilde{f}_* = \frac{1}{n} \sum_{i=1}^n L_{x_i} f_* .$$

By (i) and continuity in $C(G,\mathbb{R})$, $\tilde{f}_* \in \overline{\mathcal{L}(f)}$

$$M(\tilde{f}_*) \leq M(f_*)$$

$\forall y \in G$, there exits at least one x_i such that $x_i^{-1} y \in U$. (indeed $y \in x_i U$ for some x_i since $x_i U$ cover G).
Therefore

$$m(\tilde{f}_*) \geq \frac{n-1}{n} m(f_*) + \left(m(f_*) + \frac{v(f_*)}{2}\right)\frac{1}{n}$$

$$= m(f_*) + \frac{v(f_*)}{2n} > m(f_*).$$

But then

$$v(\tilde{f}_*) < v(f_*)$$

which contradicts the assertion that $v(f_*)$ is minimum of v.

To be able to define $I(f) = f_*(e)$, we have to check that there is only one constant function in $\overline{\mathcal{L}(f)}$. This appears to be somewhat difficult to attack head-on, but we can get there by a detour via right-invariant construction.
As above we can show that closure of

$$\mathcal{R}(f) = \left\{ \sum_{j=1}^{n} b_j R_{y_j} f \,,\ \sum b_j = 1,\ y_j \in G \,,\ b_j > 0 \right\}$$

contains a constant function. We'll show that if $f_* \in \overline{\mathcal{L}(f)}$, $g_* \in \overline{\mathcal{R}(f)}$ are constant, then $f_* = g_*$. This implies that there is only one constant function in either one.

- If $\varepsilon > 0$, let a_i, x_i, b_j, y_j s.t.

$$\| f_* - \sum_{i=1}^{m} a_i L_{x_i} f \|_\infty < \varepsilon$$

$$\| g_* - \sum_{j=1}^{n} b_j R_{y_j} f \|_\infty < \varepsilon$$

Since sup norm is translationally invariant and f_*, g_* are both left and right-invariant, we find

$$\| f_* - \sum b_j a_i R_{y_j} L_{x_i} f \|_\infty < \varepsilon$$

$$\| g_* - \sum a_i b_j L_{x_i} R_{y_j} f \|_\infty < \varepsilon$$

Since L_{x_i} and R_{y_j} commute (which is why we need to escape to right action), ~~and~~ the triangle equality implies

$$\| f_* - g_* \|_\infty < 2\varepsilon .$$

~~So we can now put define $I(f) = f_* =$ value of unique constant function in $\mathcal{X}(f)$. Left-invariance, $I(1) =$~~

So it makes sense to define $I(f) = \underline{f_*} =$ value of unique constant function in $\mathcal{X}(f)$. Left-invariance $I(L_x f) = I(f)$, normalization $I(1) = 1$, positivity $I(f) \geq 0$ if $f \geq 0$, and scale invariance $I(\lambda f) = \lambda I(f)$ are all obvious. Non-trivial is additivity

$$I(f+g) = I(f) + I(g).$$

To estabish additivity, let $\varepsilon > 0$, and a_i, x_i s.t.

$$\| I(f) - \sum_{i=1}^{m} a_i L_{x_i} f \|_\infty < \varepsilon \qquad (*)$$

Now consider $\tilde{g} = \sum_{i=1}^{m} a_i L_{x_i} g \in \mathcal{L}(g)$.
By property (i) above, we have

$$\mathcal{L}(\tilde{g}) \subset \mathcal{L}(g) \quad \text{so} \quad \overline{\mathcal{L}(\tilde{g})} \subset \overline{\mathcal{L}(g)}$$

so $\quad I(\tilde{g}) = I(g)$. So choose b_j, y_j s.t.

$$\| I(g) - \sum_{j=1}^{n} b_j L_{y_j} \tilde{g} \| < \varepsilon$$

which written out is

$$\| I(g) - \sum b_j a_i L_{y_j} L_{x_i} g \|_\infty < \varepsilon$$

On the other hand, $(*)$ implies

$$\| I(f) - \sum b_j a_i L_{y_j} L_{x_i} f \|_\infty < \varepsilon$$

So by triangle inequality

$$\| I(f) + I(g) - \sum b_j a_i L_{y_j} L_{x_i} (f+g) \| < \varepsilon$$

This implies $I(f) + I(g) \in \overline{\mathcal{L}(f+g)}$ but since $I(f+g)$ is the only constant there, we must have

$$I(f+g) = I(f) + I(g).$$

Remains to show that if $\tilde{I}$ is another continuous linear functional on $C(G,\mathbb{R})$ satisfying

$$\tilde{I}(L_x f) = \tilde{I}(f) \qquad \forall f \in C(G,\mathbb{R})$$

$$\tilde{I}(1) = 1$$

then $\tilde{I} = I$.

By linearity and translational invariance,

$$\tilde{I}\left(\sum a_i L_{x_i} f\right) = \sum a_i \tilde{I}(L_{x_i} f) = \tilde{I}(f)$$

$\forall \sum a_i L_{x_i} f \in \mathcal{X}(f)$.

By continuity $\tilde{I}(g) = \tilde{I}(f) \quad \forall g \in \overline{\mathcal{X}(f)}$

in particular

$$\tilde{I}(f) = \tilde{I}(I(f)) = \tilde{I}(I(f)\cdot 1) = I(f)\,\tilde{I}(1) = I(f)$$

Riesz representation theorem now constructs a measure for us, and proof of existence and uniqueness of Haar measure is complete.