Conformal representations in 2d

this is the write-up for the talk on the above topic. It has five sections:

1. **Introduction**
2. **Highest weight representations of the Virasoro algebra**
3. **Verma modules**
4. **Unitary highest weight representations**
5. **Other unitary representations of the Virasoro algebra**

The last section was not a part of the talk associated to this write-up. The presentation is guided by **[FMS 97]** and **[Sch 08]**. The following references are cited in the write-up:

References

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Introduction

Last time we had looked at global and infinitesimal conformal transformations in 2 euclidean dimensions:

$$\mathrm{Conf}^0(S^2) \cong PSL(2,\mathbb{C})$$
$$\text{Infinitesimal Conf}(S^2) \cong W \oplus \overline{W}$$

here $W = \langle L_n, n \in \mathbb{Z} \mid [L_n, L_m] = (n-m)L_{n+m}\rangle$ is the Witt algebra, which is a *complex* Lie algebra. An infinitesimal projective representation of the Witt algebra on a projective Hilbert space $P(\mathcal{H})$ cannot in general be lifted to a representation on the Hilbert space $\mathcal{H}$. Instead the representation of a *central extension* of W lifts to a representation on $\mathcal{H}$. The Virasoro-algebra

$$\mathrm{Vir} = \langle Z, L_n\, n \in \mathbb{Z} \mid [L_n, L_m] = (n-m)L_{n+m} + \frac{n}{12}(n^2-1)Z\delta_{n,-m}, [L_n, Z] = 0\rangle$$

is the unique central extension of the Witt algebra.

There is **another situation** I would like to investigate. In the Lorentzian setting we have (see **[Sch 08]**, chapter 2):

$$\mathrm{Conf}^0(\mathbb{R}^{1,1}) \cong \mathrm{Diffeo}_+(\mathbb{R}^1)^2$$
$$\mathrm{Conf}^0(S^{1,1}) \cong \mathrm{Diffeo}_+(S^1)^2,$$

where $\mathrm{Diffeo}_+(X)$ is the group of orientation preserving diffeomorphisms. The two groups above are actually isomorphic to one another. In contrast to the euclidean case, where the global transformations were a finite-dimensional Lie group and the local conformal transformations were infinite dimensional, here the global transformations already form an infinite dimensional Lie group:

Theorem. *If M is a compact manifold then* $\mathrm{Diffeo}(M)$ *is a Lie group modelled on the space of smooth vector fields $\mathcal{V}(M)$. The Lie algebra can be identified with the opposite algebra of $\mathcal{V}(M)$:* $\mathrm{Lie}(\mathrm{Diffeo}(M)) \cong \mathcal{V}(M)^{op}$.

Here $\mathcal{V}(M)$ is given the topology of uniform convergence of all derivatives, ie the topology generated by the pseudometrics $\sup_{x\in M} d(D^\alpha x(m), D^\alpha x'(m))$, $\alpha \in \mathbb{N}$ with D the total derivative.

A motivation for how this works can be seen via the exponential flow

$$X \mapsto (m \mapsto \exp_m(X)),$$

which for small X will be given by diffeomorphisms. A small neighbourhood of 0 will provide a chart the identity. In these definitions choices of metric on M are implicit, further details may be found in **[Mil 84]** or the online lecture notes **[Nee 15]**, since it seems to be difficult to get your hands on **[Mil 84]**.

Coming back to the conformal group, one has that $\mathcal{V}(S^1)$ is generated by

$$\frac{d}{d\theta}, \quad \cos(n\theta)\frac{d}{d\theta}, \quad \sin(n\theta)\frac{d}{d\theta}$$

(in the sense that their span is dense), which can be seen from considering the Fourier expansion of a vectorfield. Thus we can find generators of the complexification $\mathcal{V}(S^1)_{\mathbb{C}}$:

$$L_n := z^{1-n}\frac{d}{dz} \left(= -ie^{-in\theta}\frac{d}{d\theta} \right).$$

We recognise the Witt algebra. It follows that $\mathrm{Lie}(\mathrm{Conf}(S^{1,1}))_{\mathbb{C}}$ has a dense copy of $W \oplus \overline{W}$ (note that $g_{\mathbb{C}} \cong (g^{\mathrm{op}})_{\mathbb{C}}$). The following theorem shows us how the Virasoro algebra will appear also in this setting:

Remark. (**[PS 86]**) There exists a central extension $\mathcal{F}$ of $\mathrm{Diffeo}_+(S^1)$ by $U(1)$ so that

$$\mathrm{Lie}(\mathcal{F}) = \mathrm{Vir}_{\mathbb{R}},$$

where $\mathrm{Vir}_{\mathbb{R}}$ is a real form of the Virasoro algebra, that is $(\mathrm{Vir}_{\mathbb{R}})_{\mathbb{C}} \cong \mathrm{Vir}$.

Remark. (**[Lem 97]**) There does not exist a complex group G with $\mathrm{Lie}(G) = \mathrm{Vir}$.

This leads to our **concluding remarks** for the introduction. There are two basic scenarios:

1. Euclidean: The Infinitesimal conformal transformations are represented via representations of $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$.
2. Lorentzian: A projective representation of the conformal group induces a representation of $\mathrm{Vir}_{\mathbb{R}} \oplus \overline{\mathrm{Vir}_{\mathbb{R}}}$, which may be complexified to a representation of $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$.

So both settings involve the representation theory of the Virasoro algebra. In the following the second perspective will be used to motivate definitions and examples.

Highest weight representations of the Virasoro algebra

Def. A representation of Vir is a Lie algebra homomorphism $\rho: \text{Vir} \to \text{End}(V)$ where V is a complex vector space.

Def. A representation $\rho : \text{Vir} \to \text{End}(V)$ is unitary wrt a positive definite hermitian form H on V if

$$H(\rho(L_n)v, w) = H(v, \rho(L_{-n})w), \qquad H(\rho(Z)v, w) = H(v, \rho(Z)w)$$

for all $v, w \in V, n \in \mathbb{Z}$, that is if $\rho(L_n)$ and $\rho(L_{-n})$ are formal adjoints and $\rho(Z)$ is formally self-adjoint.

Remarks.

1. V need not be complete wrt H, indeed throughout the talk we will consider mainly vector spaces of countable dimension.

2. With

$$\frac{d}{d\theta} \equiv iL_0, \qquad \cos(n\theta)\frac{d}{d\theta} \equiv \frac{i}{2}(L_n + L_{-n}), \quad \sin(n\theta)\frac{d}{d\theta} \equiv -\frac{1}{2}(L_n - L_{-n})$$

   the condition of unitarity is such that these generators of the physical symmetries are formally anti-symmetric, which is a necessary condition if we want to integrate the infinitesimal representation to a unitary representation of $\mathcal{F}$ (the central extension of the conformal group).

3. One may take a more general perspective about this. One may ask that a unitary representation be a tuple (V, H, α), where $\alpha : \text{Vir} \to \text{Vir}$ is an anti-linear Lie-Algebra involution so that $H(\rho(x)v, w) = H(v, \rho(\alpha(x))w)$ for all $v, w \in V, x \in \text{Vir}$, ie it includes a notion of adjoint on the Lie-Algebra compatible with the scalar product. With this perspective it actually turns out that all unitary representations must be of the form defined above, see **[CP 87]** proposition 3.4.

4. Returning the question of lifting the representation to the Lie-Group $\mathcal{F}$, the following is true:

   Theorem. *If $\rho : \text{Vir} \to \text{End}(V)$ is a unitary highest weight (to be defined in a moment) representation then $i\rho(\text{Vir}_\mathbb{R})$ can be extended to a domain $D \subset \overline{V}$ to be by essentially self-adjoint operators.*

   This allows one to lift to a representation of the subgroup of $\mathcal{F}$ generated by one-parameter subgroups with generators in $\text{Vir}_\mathbb{R}$. This does not suffice in order to lift to a representation of $\mathcal{F}$, as this subgroup is nowhere dense in $\mathcal{F}$ despite the denseness of $\mathrm{Vir}_{\mathbb{R}}$ in the Lie-Algebra of $\mathcal{F}$. This nowhere denseness is a feature made possible by $\mathcal{F}$ being infinite dimensional. However also this can be overcome:

Theorem ([GW 85]). *With the above conditions one can extend to a projective representation of* $\mathrm{Diffeo}(S^1)$.

Example **Countably many harmonic oscillators of integer energy (boson on a string).**
Let $\mathcal{H} = \mathcal{H}_{\mathrm{harm}}^{\otimes\mathbb{N}}$ be countably many products of the harmonic oscillator Hilbert space and V the subspace with only finitely many excited modes. Denote with c_n^*, c_n the creation and annihilation operators of mode n, that is $[c_n^*, c_m] = \delta_{n,m}\mathbb{1}$ and $[c_n, c_m] = 0$. For $h > 0$ we take as Hamiltionian:

$$H = \frac{h}{2} + \sum_{n>0} n c_n^* c_n$$

and define more convenient operators on V:

$$a_n = \sqrt{c_n}, \quad a_{-n} = \sqrt{n} c_n^*, \quad a_0 = \sqrt{h}\mathbb{1}.$$

Further define:

$$\rho(L_0) = H = \frac{1}{2}a_0^2 + \sum_{k>0} a_{-k}a_k, \quad \rho(L_m) = \sum_{k,l\in\mathbb{Z},k+l=m} a_k a_l, \quad \rho(Z) = \mathbb{1}.$$

These operators send V to V and it can be checked that they satisfy the commutation relations for this to be a representation. With the scalar product on $\mathcal{H}$ this becomes a unitary representation. This representation has some more interesting properties, for example if $m > 0$ then every summand in $\rho(L_m)$ leads with annihilation operators and

$$\rho(L_m)|\mathrm{vac}\rangle = 0, \quad \rho(L_0)|\mathrm{vac}\rangle = \frac{h}{2}|\mathrm{vac}\rangle$$

and $\sum_{n\in\mathbb{N}} \rho(\mathrm{Vir})^n|\mathrm{vac}\rangle = V$.

Def. A vector $v \in V$ is called cyclic for a representation $\rho : \mathrm{Vir} \to \mathrm{End}(V)$ the sub-representation generated by it is V, that is if

$$\sum_{n\in\mathbb{N}} \rho(\mathrm{Vir})^n v = V.$$

Def. A representation $\rho : \mathrm{Vir} \to \mathrm{End}(V)$ is called a highest weight representation of weight $(c, h) \in \mathbb{C}^2$ (or a Virasoro module of weight (c, h)) if there is a cyclic $v_0 \in V$ so that:

$$\rho(Z)v_0 = cv_0$$
$$\rho(L_0)v_0 = hv_0$$
$$\rho(L_n)v_0 = 0 \text{ for } n > 0.$$

Note that thus the above example is also a highest weight representation of weight $(1, \frac{h}{2})$. For $h \geq 0$ highest weight representations are also called positive energy representations, as L_0 is often interpreted as a Hamiltonian. From the commutation relations it follows that $\rho(L_0)\rho(L_{-n_1}) \cdot ... \cdot \rho(L_{-n_k})v_0$ is an eigenstate of energy $h + \sum_i n_i \geq h$ if $n_i > 0$. Since such vectors will span the vector space $\rho(L_0)$ will have positive spectrum.

Remark. From the commutation relations it is clear that $\rho(L_n)v_0 = 0$ for all $n > 0$ is equivalent to $\rho(L_1)v_0 = 0 = \rho(L_2)v_0$.

Verma modules

We will give a definition of a Verma module with a simple formula. Afterwards we construct it in a more involved manner.

Let $\mathfrak{h} = \text{span}\{L_0, Z\}$ be the Cartan-subalgebra of Vir, $(c, h) \in \mathfrak{h}^*$ a dual element of $\mathfrak{h}$ and $\mathfrak{n}_+ = \text{span}\{L_n \mid n > 0\}$, $\mathfrak{n}_- = \text{span}\{L_n \mid\ n < 0\}$ so that we have the decomposition $\text{Vir} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$. Let $\mathfrak{U}(\mathfrak{g})$ denote the enveloping algebra of a Lie-Algebra $\mathfrak{g}$. The representation $\mathfrak{h} \to \mathbb{C}$, $aL_0 + bZ \mapsto ah + bc$ induces a representation of $\mathfrak{U}(\mathfrak{h})$. Inflate this to a representation $\mathfrak{U}(\mathfrak{h} \oplus \mathfrak{n}_+) \to \mathbb{C}$ by having $\mathfrak{n}_+$ act via 0. The Verma module $V(c, h)$ is then defined as the induced representation $\text{Ind}_{\mathfrak{U}(\mathfrak{h}\oplus\mathfrak{n}_+)}^{\mathfrak{U}(\text{Vir})}(\mathbb{C})$, which then also carries a Vir representation:

$$V(c, h) := \mathfrak{U}(\text{Vir}) \otimes_{\mathfrak{U}(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}.$$

This definition is satisfying because it allows one to extend the notion of a Verma module to more general Lie-Algebras.

Now we give a **direct construction** of this representation. Let

$$V := \mathbb{C}v_0 \oplus \bigoplus_{n=1}^{\infty} P(n), \qquad P(n) = \text{FreeVS}(\{v_\alpha \mid \alpha \text{ a descending partition of } n\}),$$

here a descending partition of n is a a tuple $(n_1, ..., n_r)$ so that $n_1 \geq ... \geq n_r$ and $\sum_{i=1}^{r} n_i = n$.

$\rho(Z)$ is taken to act as $c \cdot \mathbb{1}$ on this vector space. We define the action of the representation on $P(n)$ inductively. Start with $(n > 0)$:

$$\rho(L_{-n})v_0 = v_n, \qquad \rho(L_n)v_0 = 0, \qquad \rho(L_0)v_0 = hv_0.$$

Now if $n \geq n_1$ define:

$$\rho(L_{-n})v_{n_1,...,n_k} = v_{n,n_1,...,n_k}.$$

For $n < n_1$ the idea is to expand $v_{n_1,...,n_k}$:

$$\rho(L_{-n})v_{n_1,..,n_k} = \rho(L_{-n})\rho(L_{-n_1}) \cdot ... \cdot \rho(L_{-n_k})v_0$$

Now force the commutation relations, ie with $\rho(L_{-n})\rho(L_{-n_1}) = \rho(L_{-n_1})\rho(L_{-n}) + (-n + n_1)\rho(L_{-(n+n_1)})$ the second term now acts in the just defined manner on the vector $v_{n_2,..,n_k}$ where for the first term we continue commuting until we reach an expression for which the action is defined.

In a similar manner the action of $\rho(L_0)$ and $\rho(L_n)$ on $v_{n_1,\dots,n_k}$ is defined: commute the $\rho(L_0)$ and $\rho(L_n)$ through until you reach v_0, where the action has been defined. The result turns out to be $\rho(L_0)v_{n_1,\dots,n_k} = h + \sum_{i=1}^k n_i v_{n_1,\dots,n_k}$.

This defines $\rho(L_n)$ on a basis of V, if we extend linearly it is an inductive calculation to check that we have retrieved a representation of Vir. This representation is called the <u>**Verma module**</u> $M(c,h)$. It is a highest weight representation of weight (c,h).

Remark. For every Virasoro representation $(\tilde{\rho}, \widetilde{V})$ of weight (c,h) we have a surjective morphism of representations

$$\varphi : M(c,h) \to \widetilde{V}, \qquad v_{n_1,\dots,n_k} \mapsto \tilde{\rho}(L_{-n_1}) \cdot \ldots \cdot \tilde{\rho}(L_{-n_k})v_0,$$

ie any Virasoro module is a quotient of a Verma module. The morphism φ then induces a decomposition of $\widetilde{V}$:

$$\widetilde{V} = \mathbb{C}v_0 \oplus \bigoplus_n \varphi(P(n)).$$

From an above comment the $\varphi(P(n))$ are eigenspaces of $\rho(L_0)$ with eigenvalue $h+n$. We now show the following useful result:

Lemma. Any subrepresentation U of $\widetilde{V}$ also decomposes via φ:

$$U = \mathbb{C}\tilde{v}_0 \cap U \oplus \bigoplus_n \varphi(P(n)) \cap U.$$

Proof. Let $u \in U$, $u_i \in \varphi(P_i)$ so that $u = \sum_{i=0}^k u_i$. Note that $\rho(L_0)^s u = \sum_{i=0}^k (h+n_i)^s u_i$ must be in U again, thus

$$\begin{array}{ccccc} u & = & u_0 & +....+ & u_k \\ \rho(L_0)u & = & h\,u_0 & +....+ & (h+n_k)\,u_k \\ \vdots & = & \vdots & & \vdots \\ \rho(L_0)^{k-1}u_0 & = & h^{k-1}\,u_0 & +....+ & (h+n_k)^{k-1}\,u_k \end{array}$$

all lie U. Since the coefficient matrix $\begin{pmatrix} 1 & \dots & 1 \\ h & \dots & h+n_k \\ \vdots & & \vdots \\ h^{k-1} & \dots & (h+n_k)^{k-1} \end{pmatrix}$ is invertible (note that the columns are all independent) we get that $u_0, ..., u_k$ all lie in U. This shows the lemma.

Unitary highest weight representations

We shift our focus back to the topic of unitary highest weight representations. A trivial remark:

Remark. If c or h are not real there exists no unitary highest weight representations of weight (c, h). This follows since c, h have to be eigenvalues of the (formally self-adjoint) Z, L_0.

Lemma. Let V be a Virasoro module with real weights (c, h):

(1) There exists a unique (up to scalar multiple) hermitian form H so that

$$H(\rho(L_n)v, w) = H(v, \rho(L_{-n})w), \qquad \forall n \in \mathbb{N}, v, w \in V.$$

(2) $\ker(H)$ is the maximal proper sub-representation of V, in particular such a thing exists and $V/\ker(H)$ is an irreducible representation with a non-degenerate hermitian form.

Proof.

1: For uniqueness assume there is such a hermitian form H. The decomposition of $\mathbb{C}v_0 \oplus \bigoplus_n \varphi(P(n))$ must be an **orthogonal decomposition**, since it is a decomposition into eigenspaces of $\rho(L_0)$. Let π be the projection onto v_0, then

$$H(v_{n_1,\ldots,n_k}, w) = H(v_0, \rho(L_{n_1}) \cdot \ldots \cdot \rho(L_{n_k})w) = H(v_0, \pi(\rho(L_{n_k})w)),$$

the value of which is uniquely determined by $H(v_0, v_0)$.
To see that such a form exists one one uses the above formula to define $H(v_{n_1,\ldots,n_k}, v_{n'_1,..,n'_r})$ and see that it is real. The sesquilinear extension with the non-zero $v_{n_1,\ldots,n_k}$ as a basis will define a hermitian form on which $\rho(L_n)$ and $\rho(L_{-n})$ will be formally adjoint (the realness of $H(v_{n_1,\ldots,n_k}, v_{n'_1,\ldots,n'_r})$ is essential here).

2: It is clear that $\ker(H)$ is a sub-representation. To show that it is maximal suppose U is a sub-representation and there is a $w \in U$ with $H(w, v) \neq 0$ for some $v \in V$. We can assume v to be of the form $v_{n_1,..,n_k}$, so it follows that

$$H(w, v_{n_1,\ldots,n_k}) = H(\pi(\rho(L_{n_k})...\rho(L_{n_1})w), v_0) \neq 0$$

and $\rho(L_{n_k})...\rho(L_{n_1})w$ has a non-zero component in $\mathbb{C}v_0$. The lemma at the end of the last chapter implies that v_0 is in U. Since v_0 is cyclic this must mean that

$U = V$.

Remark. From now on we normalise $H(v_0, v_0) = 1$. By uniqueness and by all sub-representations being contained in $\ker(H)$ the (unique) hermitian H_V form on any Virasoro module V must be the pushforward of the hermitian form $H_{M(c,h)}$ on the associated Verma module $M(c, h)$ via the quotient map. The above lemma now gives some easy corollaries, because the results are nice we will call them theorems:

Theorem.

(1) $H_{M(c,h)}$ is positive semi-definite if and only iff there is a unitary Virasoro module of weight (c, h).

(2) Any unitary Virasoro module is irreducible.

(3) $M(c, h)$ is indecomposable.

(4) H_V is definite if and only if V is irreducible.

Proofs.

1: Every Virasoro module of weight (c, h) arises from quotienting out a sub-representation of $M(c, h)$, all of which are contained in $\ker(H_{M(c,h)}$. Thus $M(c, h)/\ker(H_{M(c,h)})$ is the only non-degenerate (c, h) module, on which the hermitian form is positive definite only if $H_{M(c,h)}$ is positive semi-definite.

2: $\ker(H_V)$ is the maximal sub-representation, if V is unitary this is zero and thus V is irreducible.

3: Unless $\ker(H_{M(c,h)}) = M(c, h)$ no sub-representation admits a complementary sub-representation.

4: Any sub-representation of V must live in $\ker(H_V)$.

So if we want to find the unitary Virasoro modules, we need to know when we have positive (semi-)definite hermitian forms on $M(c, h)$. Here the orthogonal decomposition $\mathbb{C}v_0 \oplus \bigoplus_n P(n)$ is useful, since we can consider the restriction of the form onto the finite-dimensional subspaces $P(n)$. As an example consider

$$H(v_0, v_0) = 1, \qquad H(v_1, v_1) = H(v_0, \rho(L_1)\rho(L_{-1}v_0) = H(v_0, 2L_0v_0) = 2h$$
$$H(v_2, v_2) = 4h + \frac{c}{2}, \qquad H(v_{1,1}, v_{1,1}) = 8h^2 + 4h, \qquad H(v_2, v_{1,1}) = 6h$$

from which we get

$$H|_{P(0)} \equiv (1), \quad H|_{P(1)} \equiv (2h), \quad H|_{P(2)} \equiv \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 8h^2 + 4h \end{pmatrix}.$$

These examples give us restrictions on which c and h allow for unitary Virasoro modules. The **Kac determinant** $A^{(n)}(c, h) := \det \left(H_{M(c,h)}|_{P(n)} \right)$ is a useful tool to investigate the definiteness properties of $H_{M(c,h)}$. For example it is clear:

$$H_{M(c,h)} \text{ non-degenerate } \iff A^{(n)}(c, h) \neq 0 \quad \forall n \in \mathbb{N}.$$

Theorem. (**[Kac 87]**)

(1) $A^{(n)}(c, h) = K_n \prod\limits_{s,r \geq 1, sr \leq n} \left(h - h_{r,s}(m(c)) \right)^{p(n-rs)}$.

(2) $A^{(n)}(c, h) > 0$ for all $c > 1$, $h > 0$, $n \in \mathbb{N}$.

For the meaning of the symbols in part (1), K_n are positive constants independent of c, h, $p(n - rs)$ gives the number of partitions of $n - rs$, and $h_{r,s}(m)$ and $m(c)$ are defined via (the evaluation is independent of the sign $\pm$ in $c(m)$):

$$h_{r,s}(m) = \frac{((m + 1)r - ms)^2 - 1}{4m(m + 1)}, \qquad c(m) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25 - c}{1 - c}}.$$

Corollary. If there is one unitary Virasoro module of weight $(c, h), c > 1, h > 0$, then all $M(c, h)$ $c > 1, h > 0$ are unitary.

This follows since the coefficients of the matrices $H_{M(c,h)}|_{P(n)}$ vary continuously with c, h, thus a switch of definiteness must result in a determinant going through zero. The theorem tells us that this cannot happen in the region $c > 1, h > 0$, thus if there is one positive definite $H_{M(c,h)}$ in this region, all of the Verma modules are unitary. The theorem also tells us that all Virasoro modules in this region are Verma modules, since non-degeneracy of the hermitian form is implied by positivity of all determinants.

Previously we had given an example of a unitary Virasoro module V_{harm} of weight $(c = 1, \frac{1}{2}h)$ by considering countably many harmonic oscillators. $V_{\text{harm}} \otimes V_{\text{harm}}$ carries a unitary representation of Vir (the tensor-product of two Lie-Algebra representation is defined via $(\rho_1 \otimes \rho_2)(L)(v \otimes w) = (\rho_1(L)v) \otimes w + v \otimes (\rho_2(L)w))$. If we look at the sub-representation generated by $|\text{vac}\rangle \otimes |\text{vac}\rangle$ we retrieve a unitary highest weight representation of weight $(1 + 1, \frac{1}{2}h + \frac{1}{2}h) = (2, h)$. Via the corollary above this gives us unitary representations in the region $c > 1, h > 0$.

The following theorem characterises all unitary Virasoro modules and concluded the talk (but not the write-up):

Theorem.

(1) $H_{M(c,h)}$ is positive definite for $c > 1, h > 0$.

(2) $H_{M(c,h)}$ is positive (semi-)definite for $c \geq 1, h \geq 0$.

(3) All positive (semi-)definite $H_{M(c,h)}$ for $c < 1, h \geq 0$ are of the form $c = 1 - \frac{6}{m(m+1)}$, $h = h_{r,s}(m)$ where $m \geq 2, 1 \leq s < r < m$ and $m, r, s \in \mathbb{N}$. $H_{M(c,h)}$ is degenerate for these values of c, h.

The proof of (3) can be found in **[GKO 85]**.

Other unitary representations of the Virasoro algebra

In the preceding section all unitary highest weight representations of Vir were constructed, and these were all irreducible. There exists another theory of lowest weight representations, in which one asks for $\rho(L_n)v_0 = 0$ for $n < 0$. Since we have a Lie-Algebra automorphism on Vir defined via $\sigma(L_n) = -L_{-n}$ that switches $\mathfrak{n}_+$ and $\mathfrak{n}_-$, the theories of lowest and highest weight representations are identical. Both are examples of finite weight representations.

Def. A representation $\rho : \mathrm{Vir} \to \mathrm{End}(V)$ is called a finite weight representation if $\rho(L_0)$ acts semi-simply on V (that is V decomposes into eigenvectors of $\rho(L_0)$) and all eigenspaces of $\rho(L_0)$ are finite-dimensional.

Remark. A consequence is that $\rho(Z)$ is proportional to the identity in an irreducible finite weight representation since it must have an eigenvalue (it preserves one of the finite dimensional eigenspaces of $\rho(L_0)$, since it commutes with everything) and its eigenspaces are sub-representations.

As an example both highest weight and lowest weight representations are finite weight representations. Another class of examples is the spaces of λ densities on S^1. Let $(\lambda, a) \in \mathbb{C}^2$ and define $W(\lambda, a) = \mathrm{FreeVS}(\{w_n \mid n \in \mathbb{Z}\})$ with an Vir action $\rho(L_k)w_n = (n + a + k\lambda)w_{n+k}$ and $\rho(Z)w_n = 0$. In this example the weight spaces all have dimension less than or equal to 1 and the representation is also irreducible if $\lambda \notin \{0, 1\}, a \notin \mathbb{Z}$. These give another class of unitary representations.

Theorem. $W(\lambda, a)$ can be given the structure of a unitary representations if and only if $\lambda \in \frac{1}{2} + i\mathbb{R}$ and $a \in \mathbb{R}$. In this case the scalar product is again unique up to a multiple.

This actually completes the classification of all finite weight unitary representations:

Theorem ([CF 88]). Let V be an irreducible unitary finite weight representation, then V is either a highest weight representation (classified in the previous sections), a lowest weight representation (admitting the same classification), or a λ-density representation $W(\lambda, a)$ with (λ, a) of the form specified above.

There are also irreducible unitary representations that are not of finite weight, this is remarked in **[CF 88]** although their reference **[Kir 81]** does not seem to be available on the internet.