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1. MOTIVATION

- Large N duality is a conjectural correspondence between the large N limit of a gauge theory with structure group $U(N)$ and a string theory

- Gopakumar-Vafa duality states that the Chern-Simons (Reshetikhin-Turaev) invariants of a link in a 3-manifold are related to the relative Gromov-Witten invariants of a 6-dimensional symplectic manifold relative to a 3-dimensional Lagrangian submanifold

Aside

- Gauge theory studies moduli spaces of connections on principal bundles
String theory studies spaces of maps from a particular class of domains into different targets:

* Both theories lead to invariants of

base manifold $\leftrightarrow$ gauge theories	C.S.
the target manifold $\leftrightarrow$ string theory	G.W.

- Chern-Simons theory is defined for real 3-manifolds
Gromov-Witten theory is defined for Calabi-Yau complex 3-manifolds

Thus, large N duality (Gopakumar-Vafa) is a correspondence between 3-dimensional real and 3 complex geometries and topologies

- To compute the duality, the most that one can do (mathematically) is to compute the Gromov-Witten invariants, the Chern-Simons invariants and compare the two answers.

- Importance of Calabi-Yau condition (physically) is that in string theory, Calabi-Yau 3-folds (6-real dim folds) provide complementary "compactified" dimensions to the 4 observed ones of the classical spacetime.

2. Comparison between Gromov-Witten and Chern-Simons free energies

2.1 "Boring" Definitions

• DEF (Eisenstein Functions)

Eisenstein functions are defined by

$$E_k(z) := \text{v.p.} \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k}$$

with Eisenstein convention:

$$\text{v.p.} \sum_{n=-\infty}^{\infty} f(n) := \lim_{N \rightarrow \infty} \sum_{n=-N}^N f(n)$$

Step in talk

* Properties Eisenstein Functions:

E.1) $E_1(z) - \pi \cot(\pi z)$ is a bounded entire function that evaluates to zero at $z = \frac{1}{2}$. Moreover is identically zero.

E.2) $E'_k(z) = -k E_{k+1}(z)$
 $\Rightarrow E_2(z) = \pi^2 \csc^2(\pi z)$
 $E_3(z) = E_1(z) E_2(z)$

E.3) Let $s(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$
 $\Rightarrow \frac{s'(z)}{s(z)} = E_1(z)$

E.4) $\frac{d}{dz} (s(z)^2 E_2(z)) = 0$

$\therefore \left[\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \right] \rightarrow \text{Write it in talk or say it in C-S free energy}$

• DEF (Bernoulli numbers B_k)

The Bernoulli numbers are defined by their generating function:

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}$$

Step in talk

* Properties of Bernoulli numbers

B.1) $\frac{z}{2} \coth\left(\frac{z}{2}\right) = \frac{z}{2} + \frac{z}{e^z - 1} \Rightarrow B_1 = -\frac{1}{2}$ and the rest of the odd Bernoulli numbers are zero.

B.2) $\frac{\exp(Nz) - 1}{z} \frac{z}{\exp(z) - 1} = N + \sum_{n=1}^{\infty} \sum_{j=1}^{N-1} j^n \frac{z^n}{n!}$

B.3) $\frac{\exp(Nz) - 1}{z} \frac{z}{\exp(z) - 1} = \sum_{n=0}^{\infty} \sum_{j=0}^N \binom{n}{j} B_j \frac{N^{n-k+1}}{n-k+1} \frac{z^n}{n!}$

B.4) $\sum_{j=1}^{N-1} j^{2p+1} = -\frac{N^{2p+1}}{2} + \sum_{k=0}^p \binom{2p+1}{2k} \frac{B_{2k} N^{2p-2k+2}}{2p-2k+2}$

$\sum_{j=1}^{N-1} j^{2p} = -\frac{N^{2p}}{2} + \sum_{k=0}^p \binom{2p+1}{2k} \frac{B_{2k} N^{2p-2k+1}}{2p+1}$

Some Bernoulli numbers

$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = \frac{1}{30}, B_6 = \frac{1}{42}$

• DEF (Riemann zeta function)

$$\zeta(z) := \frac{1}{\Gamma(z)} \int_0^\infty \frac{u^{z-1}}{e^u - 1} du ; \quad \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt$

* Relation between ζ -function and Bernoulli-numbers:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$$

* Functional equation for ζ -function (Riemann):

$$\zeta(1-k) = \frac{2}{(2\pi)^k} \cos\left(\frac{\pi k}{2}\right) \Gamma(k) \zeta(k)$$

← G. Hardy, E. Wright
Int. # theory

• The Chern-Simons partition function contains a factor that can be identified with the volume of $SU(N)$, which can be expressed in terms of the Barnes function:

• DEF (Barnes function)

$$G_2(z+1) = (2\pi)^{-z/2} e^{-(z(z+1)+\gamma)z/2} \prod_{k=1}^{\infty} \left(\left(\frac{1+z}{k} \right)^k e^{\frac{(z^2-z)}{2k}} \right)$$

where $\gamma = \lim_{n \rightarrow \infty} \sum_{k=1}^n -\ln \ln k$

* For integers $N > 1 \Rightarrow G_2(N) = \prod_{j=1}^{N-2} j!$

* Asymptotic expansion for G_2 in terms of Bernoulli numbers

$$\ln(G_2(N+1)) = \frac{N^2}{2} \ln N - \frac{3}{4} N^2 - \frac{\ln N}{12} - N \zeta'(-1) + \zeta'(-1) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}$$

(AB) ... Remember for later use

→ V. Adamchik (2004) (Barnes computation)

• The Gromov-Witten free energy require the use of polylogarithms.

• DEF (Polylogarithm functions)

$$Li_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p}$$

by inspection. $Li_1(z) = -\ln(1-z)$

$$z \frac{d}{dz} Li_{p+1}(z) = Li_p(z) \quad \rightarrow \text{important for later use}$$

* Polylogarithms indexed by positive integers have a logarithmic branch point at $z=1$.

* Polylogarithms with negative integers are meromorphic in the complex plane and relate to Eisenstein functions by a change of variables and renormalization.

* Property Polylogarithm function

Li.1) Let $g \geq 1$

$$\Rightarrow Li_{-g}(e^{-2\pi i z}) = \frac{g!}{(2\pi i)^{g+1}} \sum_{n=-\infty}^{\infty} (n+z)^{-g-1} = \frac{g!}{(2\pi i)^{g+1}} E_{g+1}(z)$$

- We will need power series expansions for $\text{Li}_{3-2g}(e^{-t})$ which are manifestly periodic with period $2\pi i$ ($g \geq 2$)
 $\Rightarrow \text{Li}_{3-2g}(e^{-t}) = (2g-3)! t^{2-2g} + \sum_{n=1}^{\infty} \frac{(2g-3)!}{(2\pi i)^{2g-2}} \left(1 + \frac{t}{2\pi i n}\right)^{2-2g} n^{2-2g}$
 $= (2g-3)! t^{2-2g} + (2g-3)! \sum_{\substack{\text{even} \\ h \geq 0}} 2 \binom{2g+h-3}{h} (-t)^h (2\pi i)^{2-2g-h} \zeta(2g+h-2)$

Away to convert a non-holomorphic function at $z=0$ into a holomorphic one

- * $\text{Li}_1(e^{-t})$ and $\text{Li}_3(e^{-t})$ have a branch point at $t=0$ and can not be expanded into a Taylor series.
 $\hookrightarrow$ Fixed by adding logarithmic "counterterms" that render them holomorphic in a neighborhood of 0

E.g. $\text{Li}_1(e^{-t}) = -\ln(1-e^{-t}) \xrightarrow{z \rightarrow 0} -\ln(t)$

$\Rightarrow \text{Li}_1(e^{-t}) + \ln(t)$ is holomorphic at $z=0$
 $\hookrightarrow$ fixed branch for the logarithm

- Lemma The polylogarithms admit the following series expansions:

(i) $\text{Li}_1(e^{-t}) + \ln t = \frac{t}{2} + \sum_{\substack{\text{even} \\ m \geq 2}} \frac{2}{m} \frac{1}{(2\pi i)^m} \zeta(m) (it)^m$

(ii) $\text{Li}_3(e^{-t}) + \frac{t^2}{2} \ln t = \zeta(3) - \zeta(2)t + \frac{3t^2}{4} + \frac{t^3}{12}$
 $- \sum_{\substack{\text{even} \\ m \geq 4}} \frac{2}{m(m-1)(m-2)} (2\pi i)^{2-m} \zeta(m-2) (it)^m$

Skip in talk

Proof

Set $f(t) = \ln(t) + \text{Li}_1(e^{-t})$ and compute

$f(t) = \ln(t) - \ln(1-e^{-t})$
 $= -\ln\left(1 - \left(\frac{t}{2} - \frac{t^2}{6} + o(t^3)\right)\right) \rightarrow \text{expanding } \text{Li}_1(e^{-t})$
 $= \text{Li}_1\left(\frac{t}{2} - \frac{t^2}{6} + o(t^3)\right)$
 $= \frac{t}{2} - \frac{t^2}{6} + o(t^3)$

$f(t)$ is holomorphic near the origin.

Compute: $f'(t) = \frac{1}{t} - \frac{t}{12} + o(t^2)$ $\frac{d}{dt}(f'(t)) = \frac{1}{2} - \frac{t}{6} + o(t)$

Now $f(t) = \sum_{n=1}^{\infty} \frac{1}{n} e^{-nt} \ln t$

$\Rightarrow \frac{d^2}{dt^2} f(t) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^k \frac{n^{k-1} k t^{k-2}}{(k-2)!}$
 $= \sum_{k=1}^{\infty} (-1)^k \frac{\zeta(1-k) k t^{k-2}}{(k-2)!}$

This allows to analytically continue $f(t)$. Integrating twice with the constants of integration given by the first terms in the power series for $f(t)$

$\Rightarrow f(t) = \frac{t}{2} + \sum_{k=1}^{\infty} (-1)^k \frac{\zeta(1-k) t^k}{k!}$

Using the functional equation

$\zeta(1+k) = 2(2\pi)^{-k} \cos\left(\frac{\pi k}{2}\right) \Gamma(k) \zeta(k);$

$\Gamma(k) = (k-1)!$ and expanding in power series $\cos\left(\frac{\pi k}{2}\right)$

and it follows the result for $\text{Li}_1(e^{-t})$

For the second equality set $g(t) = \text{Li}_2(e^{-t}) + \frac{t^2}{2} \ln t - \frac{3}{4} t^2$
 $\Rightarrow g(0) = 5(3),$

$$g'(t) = -\text{Li}_1(e^{-t}) + t \ln t - t$$

$$\Rightarrow g'(0) = 5(2)$$

and $g''(t) = \text{Li}_0(e^{-t}) + \ln t$

$$= \frac{t}{2} + \sum_{m=2, \text{even}} \frac{2}{m} \frac{1}{(2\pi)^m} 5(m) (it)^m$$

Integrating twice, using the proper constants of integration, reindexing the sum with $n = m+2$ and using the definition of $g(t)$ the result for $\text{Li}_2(e^{-t})$ follows.

2.2 Chern-Simons free energy

Recall from previous talk, the $SU(N)$ -Witten-Reshetikhin-Turaev invariant of S^3 is given by:

$$Z(S^3) = D^{-1} = \tau_k^{\text{sl}_N \mathbb{C}}(S^3) = N^{-1/2} (k+N)^{\frac{(1-N)}{2}} \prod_{j=1}^{N-1} \left[2 \sin \left(\frac{\pi j}{k+N} \right) \right]^{N-j}$$

and the Chern-Simons free energy is given by

$$F_M = \ln \left(\frac{Z^{\text{un}}(M)}{Z^{\text{var}}(M)} \right)$$

Problems:

- Formal mathematical definition of the $U(N)$ partition function has not yet been agreed upon.
- $Z^{\text{un}}(M)$ has no mathematical definition

So we define:

• DEF The unnormalized Chern-Simons free energy

$$F_M^{\text{CS}} = \ln(\tau_k^{\text{sl}_N \mathbb{C}}(M))$$

Remember

(R-T) invariants corresponding to $U_q(\text{sl}_N \mathbb{C})$ at level k is defined as

$$Z(L) = (p-q)^{\langle L, \rho \rangle} D^{-S(L_M) - S(L_M)^{-1}} F(L_M)$$

arising from TQFT and denoted by

$$Z^{\text{CS}}(M) = Z^{\text{sl}_N \mathbb{C}}(M)$$

Skip in talk

It follows that for $Z(S^3)$

$$\Rightarrow F_{S^3}^{\text{CS}} = \frac{1-N}{2} \ln(k+N) - \frac{1}{2} \ln N + \sum_{j=1}^{N-1} (N-j) \ln \left[2 \sin \left(\frac{\pi j}{k+N} \right) \right]$$

$$= \frac{1-N}{2} \ln(k+N) - \frac{1}{2} \ln N + \sum_{j=1}^{N-1} (N-j) \ln \left(\frac{2\pi j}{k+N} \right) +$$

$$+ \sum_{j=1}^{N-1} (N-j) \left(\sum_{n=1}^{\infty} \ln \left(1 - \frac{j^2}{n^2 (k+N)^2} \right) \right)$$

Barnes term

CS1

• DEF Perturbative Chern-Simons free energy (F^{pert})

$$F^{\text{pert}} = \sum_{j=1}^{N-1} (N-j) \left(\sum_{n=1}^{\infty} \ln \left(1 - \frac{j^2}{n^2 (k+N)^2} \right) \right)$$

$$= - \sum_{j=1}^{N-1} (N-j) \left(\sum_{p=1}^{\infty} \frac{x^{2p} j^{2p}}{4^p \pi^{2p}} \zeta(2p) \right)$$

where we have used $k+N = \frac{2\pi}{x}$

$$\ln(1-z) = -z + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \dots$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

* From the path integral approach to the Chern-Simons theory we expect that the free energy has a 'loop' expression of the form

$$F = \sum_g \sum_h x^{2g-2+h} N^h F_{g,h}$$

* Then, manipulating F^{pert} by using (B.4) we get

$$F^{\text{pert}} = \sum_g \sum_{\substack{\text{even} \\ h \geq 2}} \binom{2g+h-3}{h} \frac{B_{2g}}{g(2g-2)} (2\pi)^{2-2g-h} \zeta(2g-2+h) N^h x^{2g-2+h}$$

g > 1 term

g = 1 term $\leftarrow + \sum_{\substack{\text{even} \\ h \geq 2}} \frac{1}{6} (2\pi)^{-h} \zeta(h) N^h x^h \dots \text{CS2}$

g = 0 term $\leftarrow - \sum_{\substack{\text{even} \\ h \geq 4}} \frac{2}{h(h-1)(h-2)} (2\pi)^{2-h} \zeta(h-2) N^h x^{h-2}$

2.3 Gromov-Witten free-energies

• DEF (Gromov-Witten free energy)

The Gromov-Witten free energy of a Calabi-Yau 3-fold X is the following formal function depending on a complex parameter y and a cohomology class $\ell \in H^2(X; \mathbb{C})$

$$F_X^{\text{GW}}(\ell, y) := \sum_{g=0}^{\infty} \sum_{\beta} N_{g,\beta}(X) e^{-\langle \ell, \beta \rangle} y^{2g-2}$$

* The Restricted Gromov-Witten free energy is the sum over all non-zero homology classes β , denoted by $\hat{F}^{\text{GW}}$

In our case of interest $X = X_{S^3} \rightarrow$ the resolved conifold

$\Rightarrow$ The restricted Gromov-Witten free energy of X_{S^3} is

$$\hat{F}^{\text{GW}}_{X_{S^3}} = \sum_{d=1}^{\infty} \frac{1}{d} \left(2 \sin \frac{dy}{2} \right)^{-2} e^{-\ell d}$$

$\hookrightarrow$ The full Gromov-Witten energy that appears in the gauge-string duality.

* Note ℓ is assumed to be in the right half-plane for the series to converge.

* Note $\hat{F}^{\text{GW}}_{X_{S^3}} = \sum_{g=0}^{\infty} \sum_{d=1}^{\infty} N_{g,d} y^{2g-2} e^{-\ell d}$

$\hookrightarrow$ follows from definition.

- Using the relation between Bernoulli numbers and the $\csc^2(x)$ given by the Eisenstein functions and polylogarithms we can prove that

$$\begin{aligned}\hat{F}_{X_{S^3}}^{GW} &= \int_{\mathbb{R}^+} \frac{1}{t^2} \csc^2\left(\frac{dt}{2}\right) e^{-t\phi} \\ &= \sum_{g=0}^{\infty} (-1)^{g-1} (2g-1) \frac{\text{Bag Li}_{3-2g}(e^{-t})}{(2g)!} y^{2g-2} \dots \text{GW1} \\ &= \sum_{g=0}^{\infty} \hat{F}_g^{X_{S^3}} y^{2g-2}\end{aligned}$$

Skip in talk (probably, depends on time).

$$\therefore N_{g,d}(X_{S^3}) = d^{2g-3} (-1)^{g-1} (2g-1) \frac{\text{Bag}}{(2g)!}$$

↳ serves us to define the genus contribution to the restricted Gromov-Witten energy.

- Combining (GW1) with expansion of Li_{3-2g} for $g \geq 2$

$$\begin{aligned}\Rightarrow \hat{F}_g^{X_{S^3}} &= \frac{\text{Bag}}{2g(2g-2)} (it)^{2-2g} + \dots \text{GW2} \\ &+ \frac{\text{Bag}}{g(2g-2)} \sum_{\substack{\text{even} \\ h \geq 0}} \binom{2g+h-3}{h} (2\pi)^{2-2g-h} J(2g+h-2) (it)^h\end{aligned}$$

- For $g=0$ and $g=1$ we use the polylogarithm formula and Lemma

$$\hat{F}_1^{X_{S^3}} = \frac{6}{24} - \frac{1}{12} \ln 6 + \sum_{\substack{\text{even} \\ h \geq 2}} \frac{1}{6h} (2\pi)^{-h} J(h) (it)^h \dots \text{GW3}$$

$$\begin{aligned}\text{and } \hat{F}_0^{X_{S^3}} &= J(3) - J(2)t + \frac{3t^2}{4} + \frac{t^3}{12} - \frac{t^2}{2} \ln t \\ &- \sum_{\substack{\text{even} \\ h \geq 4}} \frac{2}{h(h-1)(h-2)} (2\pi)^{2-h} J(h-2) (it)^h \dots \text{GW4}\end{aligned}$$

- The degree 0 and 1 are constant maps and we must include them
- The second term in (GW2) is almost identical to (CS2)
- We can obtain the degree zero generating function by using the fact $\chi(X_{S^3}) = 2$ and

$$\frac{\text{Bag}}{g(2g-2)} (2\pi)^{2-2g} J(2g-2) = \frac{(-1)^{g-1} (2g-1) \text{Bag Bag}_{g-2} \chi(X_{S^3})}{(2g-2)! (2g)!}$$

where for $g \geq 2$ is the negative of the degree zero invariants $N_{g,0}$ for the Gromov-Witten invariants computed via localization

- Genus 0 and 1 line up as well. There are no contracted genus zero or one stable curves fixed by the torus action so $N_{0,0} = 0 = N_{1,0}$

⇒ Degree zero generating function is

$$\sum_{g=0}^{\infty} N_{g,0} y^{2-2g} = - \sum_{g=2}^{\infty} \frac{\text{Bag}}{g(2g-2)} (2\pi)^{2-2g} J(2g-2) y^{2g-2}$$

which cancels exactly the $h=0$ term.

- To avoid major discrepancies between GW and CS free energies we should include the degree zero terms
(These discrepancies take care of themselves this way).

Now we have $\vec{F}_0 = \vec{F}_0$, $F_1 = \vec{F}_1$ and

$$F_g(X_{S^3}) = \frac{B_{2g}}{2g(2g-2)} (it)^{2-2g} \dots \textcircled{\text{GW5}}$$

$$+ \frac{B_{2g}}{g(2g-2)} \sum_{\substack{\text{even} \\ h \geq 2}} \binom{2g+h-3}{h} (2\pi)^{2-2g-h} \zeta(2g+h-2) (it)^h$$

2.4 Comparison between Gromov-Witten and Chern-Simons free energies

- Combine (CS1), (CS2) and the asymptotic expansion for the Barnes term (AB)

$\Rightarrow$ the unnormalized Chern-Simons free energy is

$$F_{S^3}^{CS}(N, x) = \frac{N(N-1)}{2} \ln x + \frac{1-N}{2} \ln(k+N) + \frac{N^2}{2} \ln N$$

$$- \frac{1}{2} \ln N - 3 \frac{N^2}{4} - \frac{1}{12} \ln N - \zeta'(0)N + \zeta'(1)$$

$$- \sum_{\substack{\text{even} \\ h \geq 4}} \frac{2}{h(h-1)(h-2)} (2\pi)^{2-h} \zeta(h-2) N^h x^{h-2}$$

$$+ \sum_{\substack{\text{even} \\ h \geq 2}} \frac{1}{6h} (2\pi)^{-h} \zeta(h) N^h x^h$$

$$+ \sum_{g=2}^{\infty} \sum_{\substack{\text{even} \\ h \geq 2}} \binom{2g+h-3}{h} \frac{B_{2g}}{g(2g-2)} (2\pi)^{2-2g-h} \zeta(2g+h-2) N^h x^{2g-2+h}$$

$$+ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} N^{2-2g}$$

- Combine (GW3), (GW4), (GW5) and the definition of G-W free energy, the full Gromov-Witten free energy of the resolved conifold is

$$\mathcal{F}^{GW}(X_{S^3})(ty) = \frac{t}{24} - \frac{1}{12} \ln t + \zeta(3)y^{-2}ty^{-2} + \frac{3}{4}t^2y^{-2} +$$

$$+ \frac{t^3y^{-2}}{12} - \frac{t^2y^{-2}}{24} \ln t$$

$$- \sum_{\substack{\text{even} \\ h \geq 4}} \frac{2}{h(h-1)(h-2)} (2\pi)^{2-h} \zeta(h-2) (it)^h y^{-2}$$

$$+ \sum_{\substack{\text{even} \\ h \geq 2}} \frac{1}{6h} (2\pi)^{-h} \zeta(h) (it)^h$$

$$+ \sum_{g=2}^{\infty} \frac{B_{2g}}{g(2g-2)} \sum_{\substack{\text{even} \\ h \geq 2}} \binom{2g+h-3}{h} (2\pi)^{2-2g-h} \zeta(2g+h-2) (it)^h y^{2g-2}$$

$$+ \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} (it)^{2-2g} y^{2g-2}$$

- By inspection,
 $t \leftrightarrow Nx$, $y \leftrightarrow x$

all infinite sums match exactly! "yes"

- But, are we done?

* Analytically: we are comparing two series expansions near the origin
 $(t, y) = (0, 0)$

+) 1st Problem:

$$(t, y) = \left(\frac{2\pi i N}{k+N}, \frac{2\pi}{k+N} \right) \xrightarrow{N \rightarrow \infty} (2\pi i, 0)$$

"Solution":

The free energy is periodic in t with period $2\pi i$.

+) 2nd Problem:

We have assumed $t \in \mathbb{C}$ + $\text{Re}(t) > 0$, then the problem with analytically continuing the free energies to a punctured neighborhood of the origin is that logarithmic terms in both expressions are ambiguous.

"Solution"

We consider free energies to be just a book keeping device for the invariants on both sides of the duality.

Since logarithmic and the other mismatching terms outside the infinite sums carry no geometric information, they pose no serious problem.

THEOREM (The Gromov-Witten and Chern-Simons free energies relation):

$$\text{Re}(\tilde{F}^{\text{GW}}(X_{S^3})(iN, x) - F(S^3)(N, x)) = \frac{5}{12} \ln x + \zeta(3) x^{-2} - \frac{1}{2} \ln(2\pi) - \zeta'(-1)$$

Proof

$$\begin{aligned} \text{Re}(\tilde{F}^{\text{GW}}(X_{S^3})(iN, x) - F(S^3)(N, x)) &= \frac{N-1}{2} \ln(k+N) - \frac{(N-1)N}{2} \ln x \\ &\quad - \frac{N^2}{2} \ln N + \frac{3N^2}{4} + \frac{1}{12} \ln N + \zeta'(0)N - \zeta'(-1) - \text{Re}\left(\frac{1}{12} \ln(ixN)\right) \\ &\quad + \zeta(3)x^{-2} - \frac{3}{4} N^2 + \frac{N^2}{4} \text{Re}(\ln(ixN)) \end{aligned}$$

Using the fact that $x = \frac{2\pi}{k+N}$ to write $\ln(k+N)$ and $\zeta'(0) = -\frac{1}{2} \ln(2\pi)$ we can combine alike terms and it follows the result

Skip in talk

↓

Gopakumar-Vafa duality! (Conjecture)

Remarks

- Large N duality is said to be exact when the full free energies are equal, and said to hold at leading order when the zero genus terms agree.

* In all cases known when duality holds at leading order it holds at all orders.

• Issue

- * In the comparison theorem due to the term $\int(\beta) x^{-2}$:
 - + The duality stops holding even at level of genus zero
 - + In physics, there's been results where these free energies are in exact agreement with each other.
 - Oversight is in the use of the Polylogarithm formula versus the large N expansion for the free energies.

In future $\rightarrow$ expectation on exact correspondence with correct normalization.

Appendix A: Gromov-Witten Invariants

- Gromov-Witten invariants mathematical theory $\leftrightarrow$ string theory.
 - $\hookrightarrow$ cohomological interpretation of intersection theory.
- Motivating question:

How many degree- d parametric curves pass through the "right" number of points in a plane modulo reparametrization of the domain.

Skip in talk

- Example (space of lines with two marked points in the plane)

$$* \text{Mod}_2(\mathbb{CP}^2, 1[\mathbb{CP}']) =$$

$$= \{ u: \mathbb{CP}' \rightarrow \mathbb{CP}^2, p_1, p_2 \mid u_*[\mathbb{CP}'] = [\mathbb{CP}'], \int u = 0, p_1 \neq p_2 \in \mathbb{CP}' \} / \sim$$

where $(u, p_1, p_2) \sim (v, q_1, q_2)$ iff $\exists$ a linear transformation (Möbius transformation), say φ such that $u = v \circ \varphi$ and $\varphi(p_i) = q_i$.
holomorphic isomorphisms of $\mathbb{CP}'$

A.1 Gromov Moduli Space

- Because Gromov-Witten invariants are defined for symplectic manifolds or for projective algebraic varieties we give the following definitions:

DEF (Symplectic Manifold)

A symplectic manifold X is a (real) $2n$ -dimensional manifold with a 2-form ω s.t. $d\omega = 0$ and $(n!)^{-1} \omega^n$ is a volume form on X .

DEF (Almost complex structure)

An almost complex structure is an endomorphism of the tangent bundle $(J \in \Gamma(\text{End}(TX)))$ s.t. $J^2 = -I$.

Such is compatible with a symplectic form, ω , if the tensor defined by

$$g(X, Y) = \omega(X, JY)$$

is a Riemannian metric.

• Examples: (Standard symplectic structure on $\mathbb{C}$ and $\mathbb{CP}^1$)

Skip in talk

* The standard symplectic structure on $\mathbb{C}$ is given by

$$\omega = \frac{i}{2} dz \wedge d\bar{z}$$

* and the standard symplectic structure on $\mathbb{CP}^1$ is given by

$$\omega = \frac{i(dz \wedge d\bar{z} + dw \wedge d\bar{w})}{2(|z|^2 + |w|^2)^2}$$

* The standard complex structure on $\mathbb{C}$ (considered as a real vector space) is just multiplication by i . In the basis $\{1, i\}$ it is given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\text{or } J = \partial_y \otimes dx - \partial_x \otimes dy = i\partial_z \otimes d\bar{z} - i\partial_{\bar{z}} \otimes dz$$

* The product of two symplectic manifolds (X_k, ω_k) , $k=1,2$ is the symplectic manifold

$$(X_1 \times X_2, p_1^* \omega_1 + p_2^* \omega_2)$$

where p_k are the natural projections.

* We are interested in the resolved conifold which is given by the natural symplectic structure inherited to $\mathbb{CP}^1 \times \mathbb{CP}^1$:

DEF (The resolved conifold)

$$X_3 := \{[z_1, z_2], (w_1, w_2, w_3, w_4) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid \begin{vmatrix} w_1 & w_2 \\ z_1 & z_2 \end{vmatrix} = 0 = \begin{vmatrix} w_3 & w_4 \\ z_1 & z_2 \end{vmatrix}\}$$

which is the large N dual of S^3

• We have used the Cauchy-Riemann operator $\bar{\partial}$ which is defined on a map

Skip in talk

$$u: \Sigma \rightarrow X$$

by

$$\bar{\partial} u = \frac{1}{2} (du + J u \circ du \circ j)$$

$$\text{or } \frac{\partial u}{\partial \bar{z}} + J_S(u) \left(\frac{\partial u}{\partial z} - X_S(u) \right), \quad J \text{ an almost complex structure.}$$

• Idea for solving the motivating question:

If we consider a curve C embedded in a smooth manifold X , it can have singularities such that are arbitrarily bad, but by parametrizing them, i.e., by looking at maps

$$f: \Sigma \rightarrow X$$

$$f(\Sigma) = C$$

then we can restrict to ourselves to curves Σ with at worst nodal singularities.

DEF (A genus g , n -marked pre-stable curve) consists of the data $(C, x_1, \dots, x_n)$ where

i) C is a (possibly n -nodal) curve of arithmetic genus g (i.e., $\chi(C_C) = 1-g$)

ii) x_i are smooth points of C

Def (Stable map)

Let X be a smooth projective variety. Then a genus g , n -marked stable map into X consists of

- A genus g , pre-stable curve $(C, x_1, \dots, x_n)$
- A map $f: C \rightarrow X$ with only many finite automorphisms

where an automorphism of a map f is a map h such that the following diagram commutes

$$\begin{array}{ccc} (C, x_1, \dots, x_n) & \xrightarrow{f} & X \\ \downarrow h & \searrow f & \\ (C, x_1, \dots, x_n) & & \end{array}$$

- We want to compute

$$\bar{M}_{g,n}(X, \beta) = \{ f: (C, x_1, \dots, x_n) \rightarrow X \mid f_*[C] = \beta, f \text{ is stable} \}$$

which is a refinement of the moduli "space" by the discrete data of the homology class of the image of f . That is, fix a homology class $\beta \in H_2(X)$

Note $\bar{M}_{g,n}$ is an empty moduli space of the homology class β does not support holomorphic curves

• Hirzebruch-Riemann-Roch Formula

The expected dimension of the moduli space $\bar{M}_{g,n}(X, \beta)$ is given by the formula

$$\dim \bar{M}_{g,n}(X, \beta) = (\dim X - 3)(1-g) + \int_{\beta} c_1 T_X + n$$

Note this formula suggests our interest in Calabi-Yau 3-folds. In such case $\dim X = 3$ and $c_1 T_X = 0$

$$\therefore \dim \bar{M}_{g,0}(X, \beta) = 0 \quad \text{for Calabi-Yau 3-folds}$$

$\Rightarrow$ We should expect finitely many curves of any genus in one of these varieties, but reality is much more complex $\ddot{\smile}$

A2. Localization - Gromov-Witten invariants

- Poincaré - Duality: Let M an n -oriented closed manifold, then the k th cohomology group of M is isomorphic to the $(n-k)$ th homology group of M $\forall k \in \mathbb{Z}$

$$H^k(M) \cong H_{n-k}(M)$$

Skip in talk

- We want to use $\overline{M}_{g,n}(X, \beta)$ to count holomorphic curves in X

- * We notice that $\overline{M}_{g,n}(X, \beta)$ comes with evaluation maps, i.e. there are maps

$$\begin{aligned} \text{ev}_i: \overline{M}_{g,n}(X, \beta) &\rightarrow X \\ \text{defined by} \quad (f: (C, x_1, \dots, x_n) \rightarrow X) &\mapsto f(x_i) \end{aligned}$$

- * Consider the cohomology class $\text{ev}_i^* \delta_i$

where $\delta_i \in H^{n_i}(X)$, the Poincaré duals

Then, the Poincaré dual of this class represents the collection of maps $f: C \rightarrow X$ s.t. $f(x_i) \in V_i$, V_i subvariety of X

- + Since the ^{cup}wedge product is Poincaré dual to ^{$i=1, \dots, n$} intersection for smooth manifolds, we have that

$$\begin{aligned} \text{ev}_1^* \delta_1 \wedge \dots \wedge \text{ev}_n^* \delta_n \\ \text{represents exactly those maps} \\ f: C \rightarrow X \text{ s.t. } f(x_i) \in V_i \quad \forall i \in \{1, \dots, n\}. \end{aligned}$$

- Since the location of the points on C is arbitrary (i.e. varies over the moduli space), then

$$\int_{\overline{M}_{g,n}(X, \beta)} \text{ev}_1^* \delta_1 \wedge \dots \wedge \text{ev}_n^* \delta_n$$

is exactly the number of genus g curves in X such that they have non-zero intersection with the subvarieties V_i as desired

• Problem

To arrive at this result the moduli space must be smooth, compact, and finite dimensional, which is not generally the case.

- * "Solution" - construct a so-called "virtual-fundamental-class"

$\overline{M}_{g,n}(X, \beta)$, which is a homology class denoted by

$$[\overline{M}_{g,n}(X, \beta)]^{\text{vir}}$$

which satisfies a number of properties, but for our interest the most important one is that it is a homology class of pure dimension.

(Note $\overline{M}_{g,n}$ usually has fractional dimension since some curves cover many times the moduli space and they are counted with fractional weight).

pure dimension $\neq$ no fractional dimension

DEF (Gromov-Witten invariants)

Our first definition of the Gromov-Witten invariants:

Let $\gamma_1, \dots, \gamma_n \in H^*(X, \mathbb{Q})$, $\beta \in H_2(X; \mathbb{Z})$

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta}^X = \int [\overline{M}_{g,n}(X, \beta)]^{\text{vir}} \text{ev}_1^* \gamma_1 \wedge \dots \wedge \text{ev}_n^* \gamma_n$$

with $\text{ev}_i \rightarrow$ the evaluation maps

Remark Since $[\overline{M}_{g,n}(X, \beta)]^{\text{vir}}$ is of pure dimension equal to the expected dimension, then the Gromov-Witten invariant $\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta}^X$ is zero unless

$$\sum_{i=1}^n \deg \gamma_i = 2 + \dim \overline{M}_{g,n}(X, \beta)$$

• Intuitive interpretation of the Gromov-Witten invariants.

* $\langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta}^X$ is the number of genus g curves in the class β that intersect the cycles $\gamma_1, \dots, \gamma_n$ (varieties $V_1, \dots, V_n$ Poincaré dual to $\gamma_1, \dots, \gamma_n$)

↳ Generally fails due to the problem stated above.

• Localization (in a very informal way).

Obviously we are assuming $[\overline{M}_{g,n}(X, \beta)]^{\text{vir}}$ exists

* What does anything that we have to do now make sense?

+ All this cumbersome structure that we have introduced will help us to calculate the G-W invariants

* How?

+ Note that these moduli spaces have a number of different maps between them; one of them is

- Forgetful morphisms: (Projection)
These are "morally" given morphisms

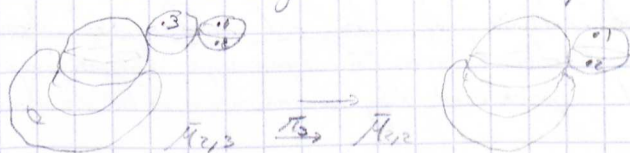
$$\pi_{n+1}: \overline{M}_{g, n+1}(X, \beta) \rightarrow \overline{M}_{g, n}(X, \beta)$$

where we map

$$(f: (C, x_1, \dots, x_{n+1}) \rightarrow X) \mapsto (f: (C, x_1, \dots, x_n) \rightarrow X)$$

i.e. we forget the $(n+1)$ marked point of the source curve.

($\pi_i \rightarrow$ forget the i -th marked point)



- To understand the following we recall that the domain of a stable map is a prestable genus g curve with marked points.

* Then, we can describe a component of the fixed point set by a graph with edges corresponding to non-contracted components and vertices corresponding to the components of the preimages of the fixed points. In addition, marked points are listed in columns under vertices:



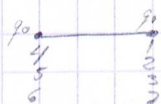
Figure App 1

Problem:

We must be careful with the forgetful morphisms, because takes an stable map into a pre-stable curve which may or may not be in the target moduli space.

Solution: is to stabilize the underlying curve.

Idea: collapse any components of the curve with too few "special" points. For example Fig App 1 will be stabilized by



→ Technique called Stable-Reduction.

- Gromov-Witten invariants satisfy the following axioms:

1. Fundamental Class Axiom:

$$\langle \gamma_1, \dots, \gamma_{n-1}, [X]^v \rangle_{g, \beta}^X = \langle \gamma_1, \dots, \gamma_{n-1} \rangle_{g, \beta}^X$$

2. Divisor Axiom: Let $\gamma_n \in H^2(X, \mathbb{Q})$

$$\Rightarrow \langle \gamma_1, \dots, \gamma_{n-1}, \gamma_n \rangle_{g, \beta}^X = \left(\int_{\beta} \gamma_n \right) \langle \gamma_1, \dots, \gamma_{n-1} \rangle_{g, \beta}^X$$

3. Point Mapping Axiom: The invariants with $\beta = 0$, $\langle \gamma_1, \dots, \gamma_n \rangle_{g, 0}^X$ satisfy

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g, 0}^X = \begin{cases} \int_X \gamma_1 \wedge \gamma_2 \wedge \gamma_3 & n=3 \\ 0 & \text{otherwise} \end{cases}$$

↳ The Gromov-Witten invariants of X include as special case the triple products in cohomology.

- A way to package the Gromov-Witten invariants in case of all the unmarked moduli spaces have zero virtual dimension (valid for Calabi-Yau 3-folds)

$$N_{g, \beta}(X) := \langle \rangle_{g, \beta}^X = \int_{[\overline{M}_{g, n}(X, \beta)]^{vir}} 1$$

Def (Gromov-Witten free energy)

of a Calabi-Yau 3-fold X is the formal function depending on a complex parameter y and a cohomology class $\ell \in H^2(X, \mathbb{C})$

$$F_X^{GW}(\ell, y) := \sum_{g=0}^{\infty} \sum_{\beta} N_{g, \beta}(X) e^{-\langle \ell, \beta \rangle} y^{2g-2}$$

Appendix B "Path-Integral" approach to C-S and G-W invariants

The intention of this appendix is to give a physical and 'superficial' description of the Chern-Simons and Gromov-Witten invariants:

• Chern-Simons Theory:

- * Chern-Simons invariants of 3-manifolds are motivated by QFT (Gauge-field theory).
- * The invariants that we discussed are the expectation values of observables of the QFT associated to the Chern-Simons action.
 - + These values do not depend on any additional geometric structure of the theory $\rightarrow$ topological invariants of the 3-manifold background. (Topological QFT's = TQFT).

Skip in Talk
Review of
previous talk

- * The fields in Chern-Simons theory are connections.

- + A connection can be viewed as a Lie algebra valued 1-form on a 3-manifold:

- Think of the sum $A = A_i \mathcal{Q}^i$ where matrices $A_i \in$ Lie Algebra and $\mathcal{Q}^i$ are 1-forms.

- If R is a representation of a Lie algebra we define a trace function on the algebra by

$$\text{Tr}_R(A) = \text{Tr}(R(A))$$

$\Rightarrow$ Chern-Simons Action of a $U(N)$ connection is given by

$$CS(A) = \frac{1}{4\pi} \int_M \text{Tr}_R(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

- * Witten suggested that the average of $CS(A)$ over all connections should be a topological invariant, since nothing in the definition depends on geometric data.

This average is called the Chern-Simons partition function.

$$Z_k(M) = \int_A e^{\frac{i}{2\pi} CS(A)} dA$$

$k \rightarrow$ string-coupling constant

- * This formal representation of the partition function is an example of a Feynman path-integral denoted by $Z(M)$

- * Note this path integral is not well-defined because the space of connections is infinite-dimensional.

- * Chern-Simons invariants of coloured framed links are the expectation values of observables constructed from holonomy of connections:
Physical language $\rightarrow$ Wilson Loop Operators

- + A coloured framed link is a framed link with a group representation associated to each component.

• Perturbative Chern-Simons theory

* Generalization of two ideas for finite dimensional integrals:

- Stationary phase approximation for oscillatory integrals
- Graphical calculus due to Feynman for evaluating Gaussian integrals

* Stationary Phase Expansion

$$\int_M e^{i\lambda H} \text{dvol}_M = \sum_{\lambda H_p=0} \left(\frac{2\pi}{\lambda} \right)^{n/2} \pi i \text{sign}(D^2 H_p) |\det D^2 H_p|^{-1/2} e^{i\lambda H(p)} + O(\lambda^{-n/2-1})$$

is exact when M is a symplectic manifold and H is $\mathbb{R}$ -invariant Hamiltonian with only non-degenerate critical points.

* Chern-Simons theory:

- invariant under action of a gauge group
- identical points are flat connections

$\Rightarrow$ Stationary phase expansion expected to hold.

* Analogy between

$$Z = \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{\hbar} \left(\frac{1}{2} Q(x,x) + \frac{1}{6} V(x,x,x) \right)}$$

$$Z_h(M) = \int_A e^{\frac{i}{8\pi\hbar} \left\{ \int_M [\text{Tr}(A \wedge dA) + \text{Tr}(\frac{2}{3} A \wedge A \wedge A)] \right\}}$$

* After some non-trivial manipulations one arrive to the conclusion that

$$F = \ln \left(\frac{Z}{Z_0} \right)$$

with

$$Z_0 = (2\pi\hbar)^{n/2} [\det Q]^{-1/2} \text{ is the free-energy}$$

and more generally

$$F = \sum_{\Gamma \in \Xi} \frac{h^{-\chi(\Gamma)}}{|\text{Aut}(\Gamma)|} W(\Gamma)$$

where $\chi(\Gamma) \rightarrow$ Euler characteristic : $2g - 2 + h = -\chi(\Gamma_2)$

$\Gamma \rightarrow$ is a trivalent graph

$\Xi \rightarrow$ set of all trivalent graphs

$W(\Gamma) \rightarrow$ contribution of any partition with graph Γ

- we can write:

* Chern-Simons free energy

$$F_M^{CS} = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} x^{2g-2+h} N^h F_{g,h}(M)$$

where $x \leftrightarrow \frac{1}{N}$ ("roleplaying")

$F_{g,h}(M) \rightarrow$ Feynman amplitude associated to the graph and 3-manifold.

* 't Hooft realized that if one "sums over all holes" one obtains a closed string with a new parameter $\epsilon = xN$

$$\Rightarrow F_M^{CS} = \sum_{g=0}^{\infty} x^{2g-2} F_g(M)$$

with $F_g(M) = \sum_{h=1}^{\infty} t^h F_{g,h}(M)$

* Summary:

- i) First step $\rightarrow$ describe Chern-Simons invariants via fat graphs which should be considered as open strings (Path-Integral).
- ii) Second step $\rightarrow$ 'sum over all holes' via geometric transition to obtain a closed string theory.

* Gramov-Witten invariants (Path-Integral approach).

* Path-integral considerations suggest that Gramov-Witten free energy is of the form $F_{GW} = \sum F_g g^{2g-2}$

* Classical equations of motion are the stationary curves of the action.
- These curves are the ones which contribute to highest order in stationary phase approximation of the partition function Z

* Note in all string theories is the existence of an extra degree of freedom s , apart from the time t :



* To discretize the path integral we have to discretize surfaces, to do so we triangulate the surfaces.

+ Note the dual graph of a triangulation is a 3-valent graph similar to a Feynman diagram
(The neighbours of these dual graphs are "flat graphs")

* Then, the partition function can be written as the matrix integral

$$Z = \int e^{-N \text{Tr}(\frac{1}{2} M^2 + \omega M^3)} dM$$

where the integral is taken over the space of all $N \times N$ Hermitian matrices

* Making a perturbative expansion we note that for each term:

- i) Each vertex contribute a factor of ωN
- ii) Each edge corresponds to a propagator and contributes a factor of N^{-1}
- iii) Each face contributes a factor of N

$\Rightarrow$ Each term in expansion has order $\omega^V N^{V-E+F} = \omega^V N^{2-2g}$

$\therefore$ Free energy is the sum of terms of order $\omega^V N^{V-E+F}$, since it's just the sum of the contributions of connected graphs, so

$$F = \sum_g F_g N^{2-2g}$$

References:

• (Main) D. Auckly, S. Koshkin "Introduction to Gopakumar-Vafa large N duality", math 0701568

• S.C.F. Rose, "Introduction to Gramov-Witten theory", arXiv 1407.1260 v1