(non-)AdS / (non-)CFT

Seminar on Holography and Large-N Dualities

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Introduction and Upshot

This is a short overview over the deformations of Maldacena's AdS/CFT correspondence [1] that were considered in the years following the original paper. A very short summary can be found in Polchinski's TASI lectures on AdS/CFT [2].

We summarize the original conjecture with the following diagram

open string spectrum dominates	$\xleftrightarrow{g_s N \ll 1}$		type IIB string theory with N coincident D3-branes		$\xleftrightarrow{g_s N \gg 1}$	closed string spectrum dominates
low energy $\downarrow$						$\downarrow$ low energy
massless open spectrum	$=$	$\mathcal{N} = 4$ $SU(N)$ SYM on $\mathbb{R}^{1,3}$		$10d$ SUGRA on $AdS_5 \times S^5$ with N units of RR 5-form flux	$=$	full closed spectrum

We will consider top-down deformations of this diagram, meaning we will give an explicit prescription on how to deform the system of branes in the string theory to modify the dual low-energy theories.

A word of warning: There will be no rigorous derivation of a dictionary, but only (physical) intuitive arguments as to what we expect (and what can, to some extend, be proven to be true more rigorously). The goal is to get some intuition on how to work with AdS/CFT. When we talk about the gauge theory on branes and the supergravity on the back-reacted geometry, we always assume the usual limits as indicated in above diagram.

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Notation

The $\mathcal{N}=4$ $SU(N)$ SYM theory living on a stack N D3-branes in flat space consists of the vector-multiplet $(A_\mu, \lambda_\alpha, \phi_i)$, $\alpha = 1,\ldots,4$ and $i = 1,\ldots,3$, where A_μ is the vector field, $\mu = 0,\ldots,3$, λ_α are 4 Weyl spinors and ϕ_i are 3 complex scalar fields. Considering the Lagrangian of this theory we can naturally decompose the multiplet and Langrangian into that of $\mathcal{N}=1$ SYM, with a vector multipet (A_μ, λ_4) and three chiral multiplets (λ_i, ϕ_i), which we will denote by Φ_i, in the adjoint of $SU(N)$. The $SU(4) = SO(6)$ R-symmetry is generically broken when adding terms that respect this $\mathcal{N}=1$ symmetry but not the $\mathcal{N}=4$.

When we talk about the scalar fields ϕ_i, we think of them as consisting of real fields A_m

$$\phi_i = \frac{A_{i+3} + iA_{i+6}}{\sqrt{2}}\,, \tag{1}$$

such that the real fields A_m, $m = 4,\ldots,9$, are $N\times N$ matrices. They appear in the massless bosonic excitation of an open string ending on a stack of N D3-branes extending in the directions $\mu = 0,\ldots,3$

$$\sum_{i,j=1}^{N}\left(\sum_{\mu=0}^{3} A_\mu^{ij}\alpha^\mu + \sum_{m=4}^{9} A_m^{ij}\alpha^m\right)|0, p_{\mathrm{NN}}, ij\rangle\,, \tag{2}$$

where α are the (bosonic) raising operators appearing in the mode expansion of the open string, p_{NN} is the momentum in the Neumann-Neumann-directions $m = 4,\ldots,9$, and where i,j are the Chan-Patton factors. One interprets the r-th eigenvalue of A_m to be the position of the r-th brane.

1 Non-coincident D3-Branes

We begin with the conceptually simplest deformation: We consider branes that are not coincident. A review on this topic can be found in [3].

It is obvious what displacing branes from (an arbitrarily chosen) origin means in the gauge theory: the chiral fields associated to the brane positions get non-vanishing, different vevs $\langle\phi_i\rangle \neq 0$. After all, after above preliminaries, this corresponds to the real matrices A_m and their eigenvalues (the brane positions) to be different in general. We choose to parametrize these 'Coulomb branch vevs', vevs of moduli in the moduli space of the $\mathcal{N}=4$ vector ($\leftrightarrow$ Coulomb) multiplet, by vevs for the chiral operators

$$\langle O^I\rangle = C^I_{i_1\ldots i_k}\,\mathrm{Tr}\,\Phi_{i_1}\ldots\Phi_{i_k}\,, \tag{3}$$

where $C^I_{i_1\ldots i_k}$ is some basis of symmetric, traceless, rank k $SO(6)$-tensors. Since we only give vevs to Φ_i, we break $\mathcal{N}=4$ to $\mathcal{N}=1$. Requiring C^I to be $SO(6)$-tensors preserves the $SU(4)$ R-symmetry (as a non-R-symmetry) after imposing the correct transformation behaviour of the vector multiplet. One can analyze this gauge theory and finds that it is non-conformal, but possesses and IR-fixed point at $\Lambda/\mu \to \infty$, where Λ is the cut-off and μ is the energy scale considered.

The SUGRA side of the theory is straightforward: We make the usual black brane ansatz for the metric

$$\mathrm{d}s^2 = H(y)^{-1/2}\mathrm{d}x^2 + H(y)^{1/2}\mathrm{d}y^2 \,, \quad \Box H = 0 \,, \tag{4}$$

where x are the coordinates along the brane, and y the ones orthogonal. The warp factor H is guessed to be a superposition of the one of a single brane stack assuming some distribution $\rho(y)$

$$\begin{aligned} H \propto & L^4 \int \mathrm{d}^6 y' \, \rho(y') \frac{1}{|y-y'|^4} \\ = & L^4 \sum_{k\geq 0, I} \int \mathrm{d}^6 y' \, \rho(y') C^I_{i_1 \dots i_k} \, y'_{i_1} \dots y'_{i_k} \frac{Y^I_k(\theta^\alpha)}{r^{k+4}} \,, \qquad r^2 = |y|^2 \,. \end{aligned} \tag{5}$$

This guess turns out to be correct after putting it into the SUGRA equations of motion. The decomposition into $SO(6)$-spherical harmonics $Y^I_k(\theta^\alpha)$ is not obvious, but it is needed to illustrate the relation to the gauge theory: The constants $C^I_{i_1 \dots i_k}$ have the same propetries as the ones appearing in the Coulomb branch vevs. This is how we relate the displaced black branes with vevs of Coulomb branch moduli in the gauge theory: By a certain choice of $\rho(y')$ we can evaluate the integral over y' to give the scalar part of expression (3). Far away from the brane insertions $r \to \infty$, the leading order term will be the one we get from a stack of N coincident black branes

$$H = \frac{L^4}{r^4} \,. \tag{6}$$

We are thus lead to the conjecture that the gauge theory with non-vanishing Coulomb branch vevs which is conformal for $\Lambda/\mu \to \infty$ is dual to the supergravity on the space described by (4) which is asymptotically, $r/L \to \infty$, $AdS_5 \times S^5$. The pattern of returning to a known duality, in this case between a conformal field theory on flat space and supergravity on $AdS_5 \times S^5$, in the limit $\Lambda/\mu \to \infty$ and $r/L \to \infty$ reappears in many known deformations. Notice how we had to preserve the $SO(6)$ symmetry on both sides by making the $SO(6)$-symmetric ansatz for the metric and by only allowing for $SO(6)$-symmetric vevs $\langle O^I \rangle$.

2 Polchinski-Strassler: Dielectric D3-Branes

Also for this example, we start by considering the gauge theory side. As proposed by Polchinski and Strassler [4], we consider a deformation to the superpotential that breaks $\mathcal{N}=4$, but preserves $\mathcal{N}=1$

$$\Delta W = \frac{1}{g_{\mathrm{YM}}} \sum m_i \mathrm{Tr}\, \Phi_i^2 \,. \tag{7}$$

This produces a theory that is non-conformal, but again possesses an IR-fixed point. Of course this will break the $SO(6)$-symmetry such that only the $U(1)$ R-symmetry and, depending on the masses m_i, a subgroup of the $SU(3)$ that rotates the Φ_i are preserved. Together with the superpotential of the original SYM theory one arrives at an F-term

$$[\Phi_i, \Phi_j] = -m_k/\sqrt{2}\epsilon_{ijk}\Phi_k \,. \tag{8}$$

We want to take another look at the $SO(6)$-symmetry. Considering instead of the perturbation above a mass term for all fermions, including the gluino λ_4 in $\mathcal{N}=1$ language,

$$m^{\alpha\beta}\lambda_\alpha\lambda_\beta + \text{h.c.}\,, \tag{9}$$

and appropriate masses m_i for scalars ϕ_i, we can preserve the $SO(6)$-symmetry but not supersymmetry. The mass matrix needs to transform appropriately in order for above term to be invariant. In fact, we can encode an $SO(6)$-invariant mass matrix (after diagonalizing) using the tensor

$$T_3 = m_1 \mathrm{d}z^1\wedge\mathrm{d}\bar{z}^2\wedge\mathrm{d}\bar{z}^3 + m_2\mathrm{d}\bar{z}^1\wedge\mathrm{d}z^2\wedge\mathrm{d}\bar{z}^3 + m_3\mathrm{d}\bar{z}^1\wedge\mathrm{d}\bar{z}^2\wedge\mathrm{d}z^3 + m_4\mathrm{d}z^1\wedge\mathrm{d}z^2\wedge\mathrm{d}z^3\,. \tag{10}$$

Here, the complex coordinates are defined via the real coordinates on the orthogonal $\mathbb{R}^6$

$$z^i = \frac{x^{i+3}+ix^{i+6}}{\sqrt{2}}\,. \tag{11}$$

This tensors turns out to be AISD: $*_6T_3 = -iT_3$. We will set $m_4 = 0$ in order to fully preserve $\mathcal{N}=1$ at the cost of breaking $SO(6)$.

We will now embed this gauge theory into a string theory setup. To do so, we will work in the simplest case $m_k = m$ for all k. Now the F-term constraint can be seen to be solved by thinking of the $N\times N$ traceless matrices Φ_i to be (generally reducible) representations of the Lie algebra $SU(2)$. We can classify all solution as direct sums of irreducible representations of dimensions $d\leq N$ such that $\sum_{d=1}^N k_d d = N$, where k_d is the number of times the d-dimensional representation appears. The gauge group $SU(N)$ is not completely broken, since we are still able to rotate blocks of the same size, such that the gauge group $(\bigotimes_d U(k_d))/U(1)$ remains. For example, consider the partition $k_d = N/d$ for some divisor d of N and all other $k_i = 0$. We then have a gauge group $SU(N/d)$ and some $U(1)$-factors.

For the sake of illustration we choose $k_N = 1$, that is to say we have an unbroken gauge group $SU(N)$. In this case the solution for the Φ_i's to the F-term is already an N-dimensional irreducible representation of $SU(2)$. We can set $A_7 = -mL_1$, $A_8 = -mL_2$ and $A_9 = -mL_3$ for the generators L_i of the representation and A_m the scalars that describe the coordinates of the branes. Something peculiar has happened: Consider the position of the branes as the eigenvalues of the operators A_m and calculate its internal radial coordinate r

$$r_0^2 \propto A_mA_m = m^2L_iL_i = m^2N^2\,, \tag{12}$$

where we used that the eigenvalue of the operator L^2 is $N(N+1) \sim N^2$. Somehow, the stack of N D3-branes now lives on a sphere of radius $r_0 \sim mN$. We did not however displace the branes by seperating them: The gauge group is still that of a single stack of branes, now however of branes of high enough dimension to not only fill the external dimensions, but also some internal submanifold.

This is an effect, that even works with a single D3-brane: A single D3-brane can be smeared out to effectively behave like a Dp-brane with $p > 3$. This was introduced by Myer [5] and can be thought of the branes becoming dielectric. Just like background electric fields can polarize neutral particles, background form-fluxes can polarize Dp-branes. One finds by carefully analyzing the CS-term of the p-brane action that there are possible couplings of unexpected dimension. One can for example write down an electric coupling of C_6 to D3-branes, which is usually only possible for D5-branes. We interpret this as a D3-brane being polarized/blown up to a D5 by background C_6-flux which is still charged under the D3. When keeping these terms and calculating the effective Lagrangian of the corresponding SYM theory on the brane one finds that the usual F-term $[\Phi_i, \Phi_j] = 0$ has to be corrected to $[\Phi_i, \Phi_j] \neq 0$, where the right-hand side is due to couplings of Φ_i to 'new' fluxes. In the case of a background C_6-potential D3's blow up to D5's with an F-term of the form we found above.

Having learned what brane-configuration we associate to the gauge theory, we can make an appropriate ansatz for the supergravity. Knowing that the $SO(6)$-symmetry of the five-sphere is broken by blown-up D3-branes, which we treat as black 5-branes, we make a less symmetric ansatz for the warp-factor in (4). This is basically the same as in the previous section, but this time we know how $\rho(y')$ will look like: The branes are distributed over a two-sphere of radius r_0 in a three-plane in the internal dimensions. In the orthogonal three-plane they are confined to the origin. So the integral in (4) reduces to an integral over the angle between y' and the position of the brane on the two-sphere. After thinking about it for a while one comes up with

$$\begin{aligned} H(w,y) =& \int_{-1}^{1} \mathrm{d}\cos\theta \, \frac{L^4}{(w^2 + y^2 + r_0^2 - 2r_0 w \cos\theta)} \\ =& \frac{L^4}{(y^2 + [w + r_0]^2)(y^2 + [w - r_0]^2)} \,, \end{aligned} \tag{13}$$

where w is the coordinate on the three-plane containting the two-sphere and y the coordinate on the orthogonal plane. In the limit $r^2 = w^2 + y^2 \to \infty$ we recover the warp factor known from the original conjecture, that is to say, far away from the brane insertions, the geometry looks like $AdS_5 \times S^5$. Note that, depending on the choice of vacuum solution to the F-term and therefore the choice of unbroken gauge group, we may have several stacks of smeared out D3-branes of different radii. Once again one can superpose the solution found here in this case.

Furthermore, one finds that the supergravity equation of motions constrain the three-form flux $G_3 = F_3 + \tau H_3$ to obey

$$H^{-1}(*_6 G_3 - iG_3) = \text{const} \propto T_3 \tag{14}$$

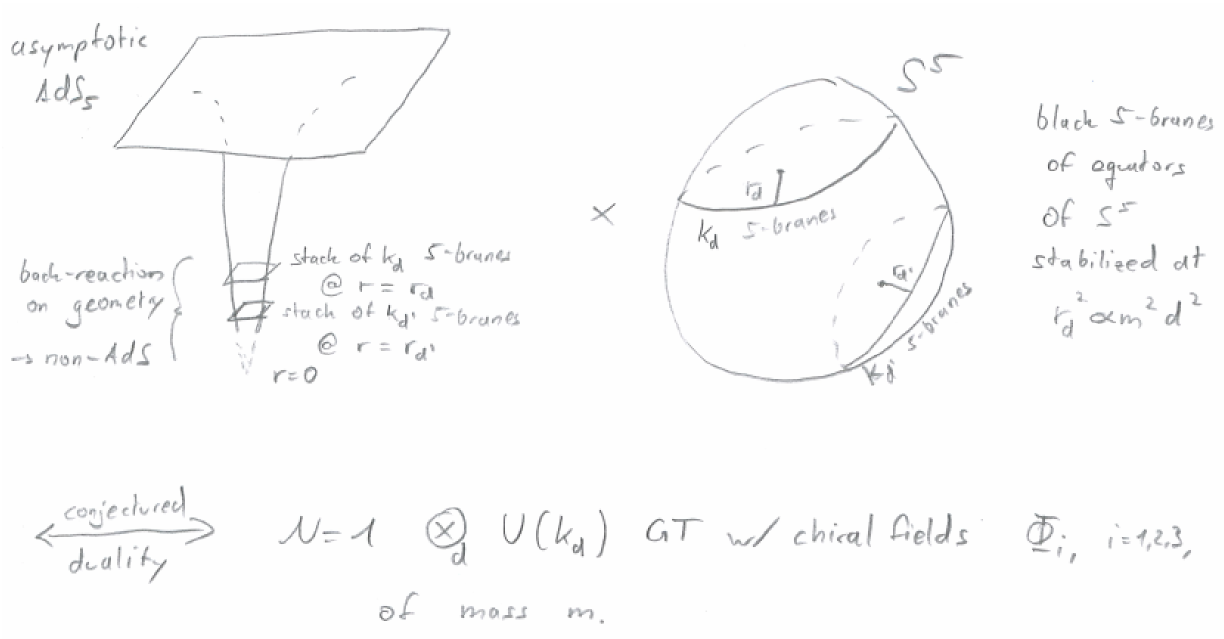

Figure 1: The Polchinski-Strassler duality (excluding fluxes on the supergravity side).

for some AISD tensor T_3. As the notation implies, the correct choice in order to recover a dictionary is to identify the constant tensor T_3 with the one that encodes the mass perturbation.

Of course, there is more to say on the supergravity solution, among other things the unexpectedly non-vanishing form-potential C_6 due to the appearence of the 'D5-branes'. We will not do so, but rather summarize the conjecture, that again has a familiar limit: The $\mathcal{N} = 4$ $SU(N)$ gauge theory perturbed by the mass term ΔW given in (7) is $\mathcal{N} = 1$ and vevs of Φ_i break the gauge group to some subgroup. There is an IR-fixed point. It is dual to supergravity on the usual geometry (4) with a warp factor as given above and the usual ISD G_3-flux is perturbed by an AISD-term proportional to a tensor T_3 that encodes the mass perturbation ΔW. In the limit $r/L \to \infty$ one asymptotically finds the geometry $AdS_5 \times S^5$. We illustrate this in figure 1.

3 Klebanov-Witten: D3-Branes in Conical Singularities

Instead of considering branes in flat space, one may consider putting branes into singularities, conical singularities to be precise. The supergravity solution to this was worked out in [6], while the gauge theory and the dictionary are discussed by Klebanov and Witten in [7, 8]. The supergravity solution requires only a slight modification of the usual ansatz (4) to

$$\mathrm{d}s^2 = H^{-1/2}\mathrm{d}x^2 + H^{1/2}\left(\mathrm{d}r^2 + r^2 g_{ij}\mathrm{d}y^i\mathrm{d}y^j\right) . \tag{15}$$

By choosing g_{ij} to be the metric of an S^5 we recover the usual ansatz. By choosing anything else, the geometry develops a conical singularity at $r = 0$, that is to say, that the internal dimensions are described by a cone $\mathbb{R}^+ \times X_5$, where X_5 is the cross-section with metric g_{ij}. The resulting warp factor is of the same form, but where the volume of the 5-sphere would enter, we of course have to insert the volume of X_5. Therefore of course, also the near-horizon limit is of the same form $AdS_5 \times X_5$. There are still N units of 5-form flux.

One possible choice of cross-section is $X_5 = T^{1,1}$. In this case the cone over X_5, called the conifold, is a CY and we have $\mathcal{N} = 1$ SUSY preserved. We can define $T^{1,1}$ of radius r as the locus

$$z_1 z_2 - z_3 z_4 = 0 , \quad r^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 , \tag{16}$$

where z_i are coordinates on $\mathbb{C}^4$. One can check that these equations are solved by parametrizing

$$z_1 = A_1 B_1 ,\ z_2 = A_2 B_2 ,\ z_3 = A_1 B_2 ,\ z_4 = A_2 B_1 , \tag{17}$$

once we impose the normalization

$$r^2 = \left(|A_1|^2 + |A_2|^2\right)\left(|B_1|^2 + |B_2|^2\right) . \tag{18}$$

There is a $U(1)$ and scaling redundancy in this description: Transforming

$$\begin{aligned} A_i \to& e^{i\alpha} A_i ,\ B_i \to e^{-i\alpha} B_i , \\ A_i \to& \lambda A_i ,\ B_i \to \lambda^{-1} B_i , \end{aligned} \tag{19}$$

leaves a solution z_i invariant. After modding out scaling by setting

$$r = |A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 , \tag{20}$$

we realize that we have a transitive action of $SU(2) \times SU(2)$ on the solutions written as a pair of $\mathbb{C}^2$-vectors (A_1, A_2), (B_1, B_2). Therefore, after also modding out the redundant $U(1)$, we can describe $T^{1,1}$ as the homogeneous space $\frac{SU(2)\times SU(2)}{U(1)}$.

We now consider the gauge theory consisting of $\mathcal{N} = 1$ chiral fields A_i and B_j in the $(N, \bar{N})$ and $(\bar{N}, N)$ of $U(N) \times U(N)$ with charges under the abelian subgroup $U(1) \times U(1)$ being $(1, -1)$ and $(-1, 1)$ respectively. The theory has a $SU(2) \times SU(2)$ symmetry acting on the vectors (A_1, A_2),(B_1, B_2). As long as there is no superpotential, the F-term is trivial, while the D-term reads

$$D = \left(|A_1|^2 + |A_2|^2\right) - \left(|B_1|^2 + |B_2|^2\right) = 0 , \tag{21}$$

which is solve by (20) for any positive r. Since a diagonal $U(1)$ subgroup acts trivially, the space of vacuum solutions can be written as $\mathbb{R}^+ \times \frac{SU(2)\times SU(2)}{U(1)}$, where the first factor comes from the choice of r/D-term solution and the second one from the global symmetries.

While this already looks very familiar, it is indeed the conifold described above, the gauge group $U(N)^2$ is too big for a stack of N branes. We expect the matrices A_i and B_j to be multiples of the identity matrix, since they should only have a single eigenvalue that describes N coincident branes. One can check that setting A_i and B_j to be multiples of $\mathbb{1}$ breaks a diagonal $U(N)$-subgroup, such that the correct gauge group remains. Furthermore we can introduce the most general superpotential that preservers both the global symmetry $SU(2) \times SU(2)$ and the R-symmetry $U(1)_R$

$$W = \lambda \epsilon^{ij} \epsilon^{kl} \mathrm{Tr}\ A_i B_k A_j B_l \,, \tag{22}$$

for some constant λ. It identically vanishes for commuting matrices A_i and B_j as in the case just discussed. We can check, that for a generic vevs of A_i and B_j, the gauge group is broken to $U(1)^N$, the expected gauge group of N non-coincident branes.

With this we again want to end this section with the conjecture: Supergravity on $AdS_5 \times T^{1,1}$ with N units of RR 5-form flux is dual to $\mathcal{N} = 1$ (spontaneously broken) $U(N) \times U(N)$ SYM on $\mathbb{R}^{1,3}$ with chiral fields A_i and B_j in the $(N, \bar{N})$ and $(\bar{N}, N)$ respectively with the superpotential (22). As a part of the dictionary, we again identify the radial coordinate r with the inverse energy scale $1/\mu$. In this example we have unbroken conformal symmetry of the cone $r \to \lambda r$ and the gauge theory is a CFT.

Since we will need the following to motivate the correspondence of the next section, we want to highlight part of the dictionary of the Klebanov-Witten duality: The two complexified gauge couplings τ_i of the $U(N)$'s are related to the axio-dilaton τ_{IIB} and Wilson loops $\int C_2$, $\int B_2$ on the two-cycle of $T^{1,1}(= S^2 \times S^3$ topologically) via

$$\tau_1 + \tau_2 = \tau_{\mathrm{IIB}} \,, \quad \tau_1 - \tau_2 = \int (C_2 + \tau_{\mathrm{IIB}} B_2) \ . \tag{23}$$

4 Klebanov-Strassler: Fractional D3-Branes in Conical Singularities

We want to introduce fractional D3-branes to the setup of the previous section. In our case, we can think of them as D5-branes that wrap the two-cycle of $T^{1,1}$ that collapses at the apex $r = 0$, such that their world-volume effectively becomes that of a 3-brane [9]. The gauge group on a stack of N D3-branes and M fractional D3-branes has been worked out in [10] and is $SU(N+M) \times SU(N)$. Compare this to the gauge group of the previous section, which looks (up to $U(1)$'s) very similar with one factor being enhanced by the fractional branes. Before turning to the full supergravity solution, we consider the limit $M \ll N$, which is know as the Klebanov-Tseytlin solution [11]. We treat the fractional branes as a perturbation that does not back-react on the internal geometry. The fluxes however, necessarily being quantized, now fulfill

$$\int_{T^{1,1}} F_5 \propto N \,, \quad \int_{S^3} F_3 \propto M \,, \tag{24}$$

where we integrate the fluxes F_{8-p} over the space surrounding the Dp-branes. While the solution of F_5 takes the familiar form proportional to the volume form on $T^{1,1}$, we now have non-vanishing $F_3 = M\omega_3$, with ω_3 the volume form on S^3, due to the (effective) presence of M D5's. To simplify, we assume the axio-dilaton to be constant. This will lead to a consistent solution, therefore a posteriori justifying the assumption. In this case, as can be read off from the type IIB SUGRA equations of motion, the 3-form fluxes fulfill

$$H_3 = g_s *_6 F_3 \,, \tag{25}$$

such that we can find the NSNS-flux

$$H_3 \propto \frac{g_s M}{r} \mathrm{d}r \wedge \omega_2 \qquad \Rightarrow \quad \int_{S^2} B_2 = g_s M \log(r/r_0) \,, \tag{26}$$

where ω_2 is the volume form of the two-sphere of $T^{1,1}$ and r_0 is some integration constant.

The fluxes govern the form of the warp factor $H(y)$ in (15). The calculation is as usual, but we have to be careful to use the gauge invariant five-form $\tilde{F}_5 = F_5 - B_2 \wedge F_3$, which has the flux number

$$\int_{T^{1,1}} \tilde{F}_5 \propto N + g_s M^2 \log(r/r_0) \equiv N_{\text{eff}}(r) \,. \tag{27}$$

This leads to, easily enough, the warp factor having the usual form, but with N replaced by N_{eff}

$$H(r) \propto \frac{g_s N_{\text{eff}}(r)}{r^4} \,. \tag{28}$$

Due to the logarithm, the geometry possess a naked singularity at some finite $r = r_s > 0$, where $H(r) \propto \log(r/r_s)$.

The Wilson loop $\int B_2$ possesses an important property in supergravity: It is 2π-periodic. Looking at (26), this now leads to the conjecture, that physics at different radial coordinates should be the same

$$\int B_2 \sim \int B_2 - 2\pi \leftrightarrow \log(r/r_0) \sim \log(r/r_0) - \frac{2\pi}{g_s M} \leftrightarrow r \sim r - r_0 e^{-\frac{1}{g_s M}} \,. \tag{29}$$

In terms of N_{eff} this equivalence corresponds to $N_{\text{eff}} \sim N_{\text{eff}} - M$. While this is not too surprising, the cone is conformally symmetric after all, we will make sense of this by considering the dual gauge theory.

As stated in the beginning, we expect the dual gauge theory to be $\mathcal{N} = 1$ $SU(N+M) \times SU(N)$ SYM coupled to chiral fields as in the previous section. The relevant 5-form flux number N_{eff} is however not integer and r-dependent. So we start by proposing that, at a given r such that $N_{\text{eff}}(r) \in \mathbb{Z}$, the supergravity solution is dual to SYM with gauge group $SU(N_{\text{eff}} + M) \times SU(N_{\text{eff}})$. If we furthermore trust that the Klebanov-Witten dictionary can be extended, we expect running couplings (compare the real parts in (23))

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} \propto \frac{1}{g_s} \,, \quad \frac{1}{g_1^2} - \frac{1}{g_2^2} \propto \frac{1}{g_s} \int B_2 \propto M \log(r/r_0) \,. \tag{30}$$

Again identifying the radial coordinate r/r_0 with the inverse energy scale Λ/μ, we expect divergent couplings g_i at some energy scales. Indeed, we can confirm this by a pure gauge theory calculation

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \propto \log(\Lambda/\mu)\,. \tag{31}$$

There is a known duality, Seiberg duality, that relates $SU(N_c)$ in the strong coupling regime with $SU(N_f - N_c)$ in the weak coupling regime. Let us see, what this means in our case: In the factor $SU(N_{\text{eff}} + M)$ we have $N_c = N_{\text{eff}} + M$ colors and $N_f = 2N_{\text{eff}}$ chiral fermions A and B. We therefore have a strong-weak duality $SU(N_{\text{eff}} + M) \leftrightarrow SU(N_{\text{eff}} - M)$. The same counting in the second factor leads to the duality $SU(N_{\text{eff}}) \leftrightarrow SU(N_{\text{eff}})$. When also exchanging the order of the two factors under the duality, we can write it as

$$SU(N_{\text{eff}} + M) \times SU(N_{\text{eff}}) \leftrightarrow SU(N_{\text{eff}}) \times SU(N_{\text{eff}} - M) \quad \text{or} \quad N_{\text{eff}} \leftrightarrow N_{\text{eff}} - M\,. \tag{32}$$

The duality takes the form we guessed on the supergravity side!

This so-called duality cascade cannot go on forever: At some point we run into the singularity on the supergravity side, since in each step, we go deeper into the AdS-space. On the gauge theory side, the duality can only make sense as long as the number of colors after dualizing is still positive.

This problem was resolved by Klebanov and Strassler [12] by considering what happens when we no longer have $M/N_{\text{eff}} \ll 1$. We start on the gauge theory side. Say we end up with the gauge group $SU(M)$" $\times SU(1)$" $= SU(M)$ after N/M duality steps. We now deal with SQCD, for which the there is a known non-perturbative correction to the superpotential. The F-term condition now enforces

$$\det_{i,j} A_i B_j = \epsilon^2 \neq 0\,, \tag{33}$$

while without the SQCD correction this vanished. This correction only appears in the $SU(M)$-theory, where A_i and B_j are now the chiral fields of this theory in the M and $\bar{M}$. The duality cascade truncates.

We want to rewrite the conifold equation (16) as

$$\det_{i,j} w_{ij} = 0\,, \quad w_{ij} = \sum_m \sigma^m_{ij} w_m\,, \tag{34}$$

where σ^m are the Pauli matrices, $\sigma^4 = i\mathbb{1}$, and w_m are complex coordinates related to z_m used before by a simple linear coordinate transformation. By comparing (33) and (34) we are lead to demand

$$\det_{i,j} w_{ij} = \epsilon^2 \neq 0\,, \tag{35}$$

and we would be correct in doing so! The geometry defined by replacing the embedding equation (16) by this one is the deformed conifold. Far away from from $r^2 = \sum_i |w_i|^2 = \epsilon^2$ it looks like the conifold, but at the apex there is a three-sphere of finite volume $\propto \epsilon^2$ such that the conical singularity is smoothed. This is just what we need to get rid of the

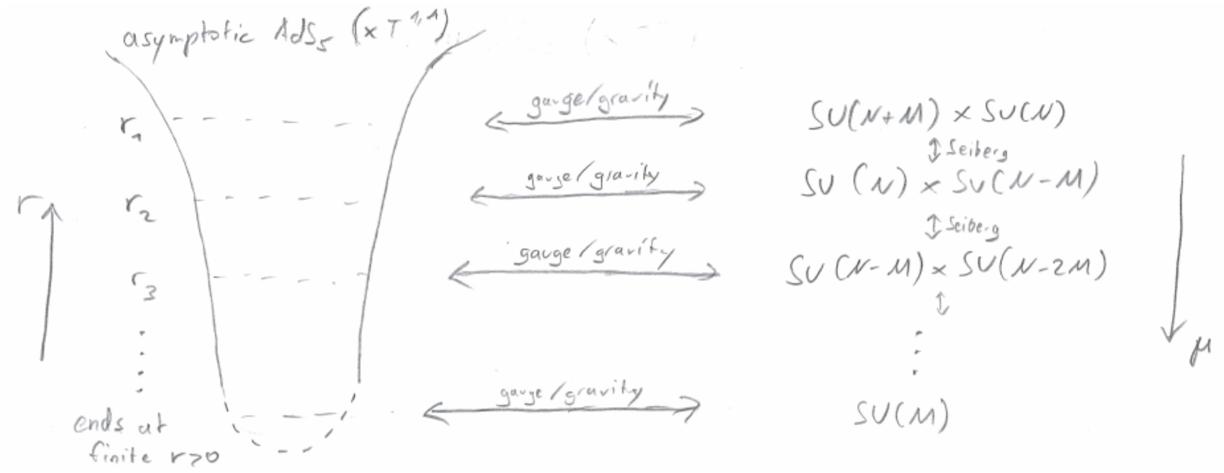

Figure 2: The Klebanov-Strassler geometry and the dual gauge groups at a given radial coordinate.

naked singularity described above and to make the F_3-flux density $|F_3|^2 \sim M^2/\mathrm{Vol}(S^3)$ finite everywhere.

Having calculated (or rather guessed) the back-reaction on the geometry, we can check that there are analytical solutions for to the equations of motion for the fluxes F_3, H_3 and F_5 and the warp factor $H(r)$ for all radial coordinates, including $r \to \epsilon$. In the limit $r/\epsilon \to \infty$ one recovers the asymptotic solution of Klebanov and Tseytlin. The warped internal geometry is widely known as the Klebanov-Strassler throat.

We again close with the conjecture: There is a cascade of dualities between the back-reacted geometry of black fractional 3-branes on the conifold, the (warped) deformed conifold, and the $\mathcal{N} = 1$ SYM theories with gauge group $SU(N + M) \times SU(N)$. The cascade on the supergravity side is realized by going down into the throat to smaller radial coordinates, while on the gauge theory side we have Seiberg duality, which reduces the number of colors by M, whenever the gauge couplings diverge. The cascade ends, when the number of colors can no longer be reduced $N_{\mathrm{eff}} \sim M$ and when we reach $r \sim \epsilon$. The duality is illustrated in figure 2.

5 D3-Branes in Orbifold-Singularities

Finally, we want to shortly consider what was historically one of the first known deformations [13]: Introducing an O-plane in the string theory (in an appropriate way, such that the D3's are in the orbifold singularity/fixed point) breaks supersymmetry to $\mathcal{N} = 2$ and results in the supergravity theory living on the geometry $AdS \times S^5/\Gamma$, where Γ is some finite group, and the gauge theory only containing the Γ-invariant fields, while the others are projected out. If we take for example $\Gamma = \mathbb{Z}_k$, the $SU(N)$ is broken to $SU(N/k)^k$ with the matter being in the bifundamental. More generally, when also adding fractional branes

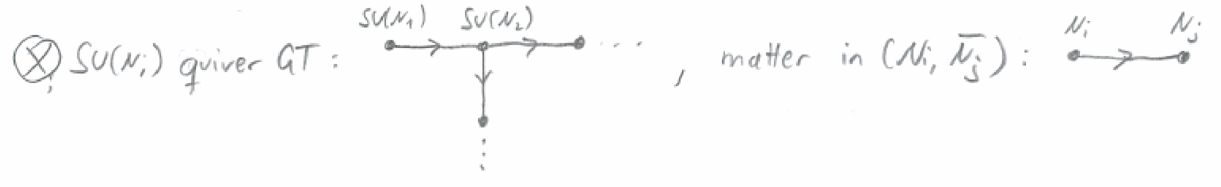

Figure 3: An illustration of a quiver gauge theory.

as in the previous section, we can achieve a gauge theory with different gauge group factors

$$SU(N) \to SU(N_1) \times SU(N_2) \times \ldots \times SU(N_k)\,. \tag{36}$$

The gauge theory is a quiver gauge theory with matter in bifundamentals $(N_i, \bar{N}_j)$, see figure 3.

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